Diagrammatic (∞, n)-categories

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The landscape of models

GEOMETRIC

Strict

INCOMPLETE

Segal
(Rezk, Tamsamani-Simpson)

Type-Theoretic

Shaped
(Complicial, Cubical)

Globular
(Grothendieck-Maltsiniotis, Batanin-Leinster)

Associative
/Quasistrict

ALGEBRAIC

The landscape of models

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BFM
BM's

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ALGEBRAIC

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The landscape of models

GEOMETRIC

Segal
(Rezk, Tamsamani-Simpson)

LOUBATON

Shaped
(Complicial, Cubical)

Strict

INCOMPLETE

feels like...

DIAGRAMMATIC

is...

bridge to?

enables diagrammatic reasoning like...

Associative
/Quasistrict

Type-Theoretic

BFM
BMS

Globular
(Grothendieck-Maltsiniotis, Batanin-Leinster)

ALGEBRAIC

A model that "just works"

- No extra data beyond cells + face & degeneracy maps
- Duals, suspensions, Gray products, joins all defined representably
- Diagrammatic arguments either just work, or can be tweaked*

*unless actually unsound

The original sin of strict n-categories

Is mixing pasting ...

$$\left(x \xrightarrow{f} y \;,\; y \xrightarrow{g} z \right) \longmapsto x \xrightarrow{f} y \xrightarrow{g} z$$

... with composition

$$\left(x \xrightarrow{f} y \;,\; y \xrightarrow{g} z \right) \longmapsto x \xrightarrow{g \circ f} z$$

The original sin of strict n-categories

To compose $f : O^n \longrightarrow X$
$g : O^m \longrightarrow X$ at the k-boundary :

① Construct the pushout

PASTE

$$\begin{tikzcd}
 & O^k \arrow[dl] \arrow[dr, hook] & \\
O^n \arrow[dr, hook] \arrow[ddr, bend right, "f"'] & & O^m \arrow[dl] \arrow[ddl, bend left, "g"] \\
 & O^n \#_k O^m \arrow[u, phantom, "\wedge"] \arrow[d, dashed, "f \#_k g"'] & \\
 & X &
\end{tikzcd}$$

in $\omega\mathrm{Cat}/X$

The original sin of strict n-categories

To compose $f : O^n \longrightarrow X$, $g : O^m \longrightarrow X$ at the k-boundary:

① Construct the pushout

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& O^n \#_k O^m \arrow[d, dashed, "f \#_k g"'] & \\
& X &
\end{tikzcd}$$

in $\omega\mathrm{Cat}/X$

② Pull back along a canonical functor

$$O^{\max(n,m)} \xrightarrow{\langle - \rangle} O^n \#_k O^m \xrightarrow{f \#_k g} X$$

COMPOSE

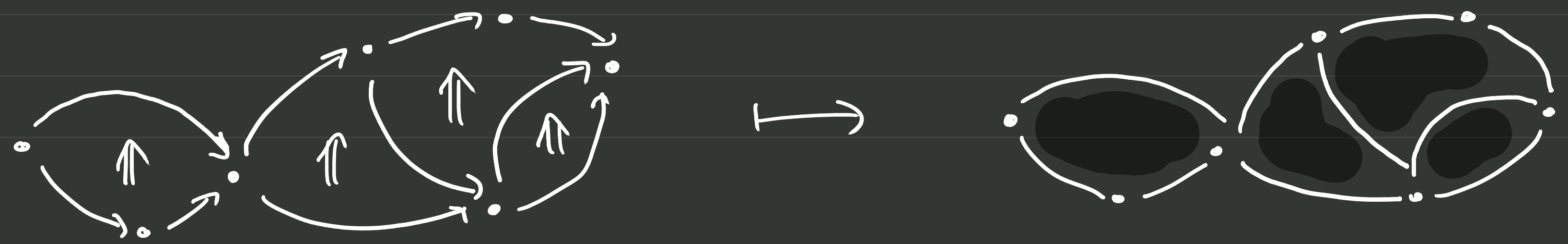
The original sin of strict n-categories

The operation of pasting is topologically sound ...

closed n-ball

$$\begin{tikzcd}[column sep=small] & O^k \arrow[dl] \arrow[dr, hook] & & & & & D^k \arrow[dl] \arrow[dr, hook] & \\ O^n \arrow[dr, hook] & & O^m \arrow[dl] & \longmapsto & & D^n \arrow[dr, hook] & & D^m \arrow[dl] \\ & O^n \#_k O^m \arrow[u, phantom, "\ulcorner"] & & & & & D^n \sqcup_{D^k} D^m \arrow[u, phantom, "\ulcorner"] & \end{tikzcd}$$

... even beyond "globular pasting diagrams" ( )



The original sin of strict n-categories

The equations of strict ω-categories are sound for the pasting of "directed cells" in regular CW complexes

Q: Are they also "complete"?

A: NO! First counterexample in dim 4!

The original sin of strict n-categories

OTOH, composition functors are "topologically unsound" (bc. of "strict Eckmann-Hilton")

The original sin of strict n-categories

OTOH, composition functors are "topologically unsound" (bc. of "strict Eckmann-Hilton")

TO RECAP:

TWO SEPARATE, INDEPENDENT PROBLEMS!

Strict ω-categories are

- sound, but incomplete for pasting diagrams,
- unsound for composing them

LAYER 1

Pasting

$\rightsquigarrow$

Regular Directed Complexes

A.H., Combinatorics of Higher-Categorical Diagrams

2024, arXiv: 2404.07273

Regular directed complexes

A regular directed complex is given by data of

1. a graded poset $P = \bigcup_{n \in \mathbb{N}} P_n$
2. an orientation $\Delta x = \Delta^+ x + \Delta^- x$

OUTPUT/TARGET

INPUT/SOURCE

FACES = covered els.

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OUTPUT/TARGET

FACES = covered els.

INPUT/SOURCE

and has the following properties:

NOT A DEF.

1. P is the face poset of a regular CW complex
2. there is an ω-category Mol/P with a basis in bijection with P

Regular directed complexes

$$\begin{tikzcd}
 & y \arrow[dr, "h"] & \\
x \arrow[ur, "g"] \arrow[rr, "f"'] \arrow[rr, phantom, shift left=4, "\Uparrow\alpha"] & & z
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \alpha \arrow[d] \arrow[dr] & \\
f \arrow[ur] \arrow[drr] & g \arrow[d] & h \arrow[d] \\
x \arrow[u] \arrow[ur] & y \arrow[ur] & z
\end{tikzcd}$$

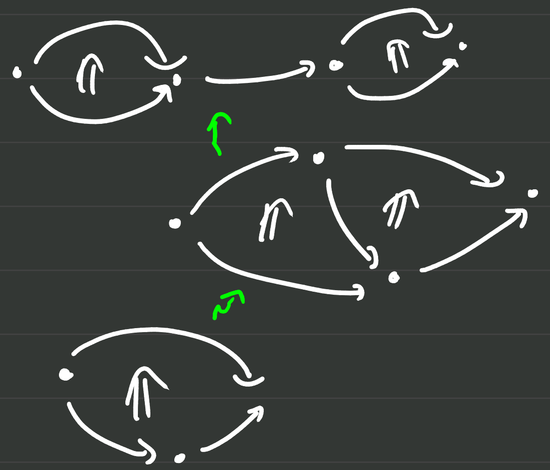
There is a category $\mathsf{RDCpx}_{\downarrow}$ whose morphisms can be interpreted at once as

- cellular maps of CW complexes,
- functors of strict ω-categories

ACTUALLY A CHARACTERISATION!

Regular directed complexes

	SPACES	ω-CATS
RDCPXs	Regular CW cpxs	
$\cup$		
MOLECULES		Composable diagrams
$\cup$		
ROUND MOLECULES	CW Balls	
$\cup$		
ATOMS	Single top-dim cell	



LOGICALLY THESE COME FIRST

LAYER 2

Directed Cell Complexes

$\rightsquigarrow$

Diagrammatic Sets

C.C., A.H., *Diagrammatic Sets as a Model of Homotopy Types*
2024, arXiv: 2407.06285

The atom category $\odot$

Objects: Atoms $\equiv$ RDCPX with greatest el.

Morphisms: <u>Cartesian</u> maps

$\equiv$ maps of RDCPX that are Grothendieck fibrations of the underlying posets

The atom category $\odot$

$\odot$ is an Eilenberg-Zilber category:

- maps $U \xrightarrow{f} V$ factor uniquely as a collapse $U \xrightarrow{p_f} f(U)$ followed by an inclusion $f(U) \xhookrightarrow{\iota_f} V$
- in a presheaf $\odot^{op} \xrightarrow{X} \underline{Set}$, every element $u \in X(U)$ factors uniquely as $U \xrightarrow{p_u} V \xrightarrow{v} X$, v non-degenerate

The atom category $\odot$

DIAGRAMMATIC SET

a presheaf $\odot^{op} \xrightarrow{X} \underline{Set}$

The atom category $\odot$

$\odot$ is a strict test category:

There is a canonical model structure on $\odot\underline{\mathrm{Set}}$, and a Quillen equivalence

$$\begin{tikzcd} \odot\underline{\mathrm{Set}} \arrow[r, bend left, "Sd_{\odot}"] \arrow[r, phantom, "\perp"] & s\underline{\mathrm{Set}} \arrow[l, bend left, "Ex_{\odot}"] \end{tikzcd}$$

with the classical model structure on $s\underline{\mathrm{Set}}$

The atom category $\odot$

$\odot$ contains Δ as a full subcategory:

And this determines a second Quillen equivalence with $s\underline{\mathrm{Set}}$!

$$\begin{tikzcd}
s\underline{\mathrm{Set}} \arrow[r, bend left, "\iota_\Delta"] \arrow[r, phantom, "\perp"] & \odot\underline{\mathrm{Set}} \arrow[l, bend left, "-_\Delta"] \arrow[r, bend left, "\mathrm{Sd}_\odot"] \arrow[r, phantom, "\perp"] & s\underline{\mathrm{Set}} \arrow[l, bend left, "\mathrm{Ex}_\odot"]
\end{tikzcd}$$

The atom category $\odot$

$\odot$ is closed under Gray products . . .

The atom category $\odot$

$\odot$ is closed under Gray products ...

... and joins ...

The atom category $\odot$

$\odot$ is closed under Gray products . . .

. . . and joins . . .

. . . and suspensions . . .

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... and joins ...

... and suspensions ...

... and duals ...

The atom category $\odot$

$\odot$ is closed under Gray products . . .

. . . and joins . . .

. . . and suspensions . . .

. . . and duals . . .

which makes it breezy to extend all these operations to diagrammatic sets.

The atom category $\odot$

The embedding $\odot \hookrightarrow \odot\underline{\mathrm{Set}}$ factors through

$$\odot \hookrightarrow \underline{\mathrm{RDCpx}}$$

rdcpxs & cartesian maps

The essential image of $\underline{\mathrm{RDCpx}} \hookrightarrow \odot\underline{\mathrm{Set}}$

is the "regular diagrammatic sets":

if $u \in nd(X)$, $\quad u : U \longrightarrow X \quad$ is mono

NON-DEGENERATE CELLS

LAYER 3

Homotopies

$\rightsquigarrow$

Coinductive Weak Invertibility

C.C., A.H., *Equivalences in Diagrammatic Sets*

2024, arXiv: 2410.00123

Equivalences in diagrammatic sets

Let U be a rdcpx, X a diagrammatic set.

A morphism $u : U \longrightarrow X$ is ...

- a pasting diagram if U is a molecule;
- a round diagram if U is a round molecule;
- a cell if U is an atom.

Yoneda: Same as $u \in X(U)$!

Equivalences in diagrammatic sets

Pasting diagrams can be pasted!

$$
\begin{tikzcd}
 & \partial^+_k U \simeq \partial^-_k V \arrow[dl, hook'] \arrow[dr, hook] & \\
U \arrow[dr, hook] \arrow[ddr, bend right, "u"'] & & V \arrow[dl, hook'] \arrow[ddl, bend left, "v"] \\
 & U \#_k V \arrow[u, phantom, "\wedge"] \arrow[d, dashed, "u \#_k v"] & \\
 & X &
\end{tikzcd}
$$

Equivalences in diagrammatic sets

Pasting diagrams can be pasted!

$$\begin{tikzcd}
& \partial^+_k U \simeq \partial^-_k V \arrow[dl, hook'] \arrow[dr, hook] \arrow[dd, phantom, "\wedge" very near end] & \\
U \arrow[dr, hook] \arrow[ddr, bend right, "u"'] & & V \arrow[dl] \arrow[ddl, bend left, "v"] \\
& U \#_k V \arrow[d, dashed, "u \#_k v"] & \\
& X &
\end{tikzcd}$$

But there is no composition

— no way to reduce a pasting diagram to a single cell.

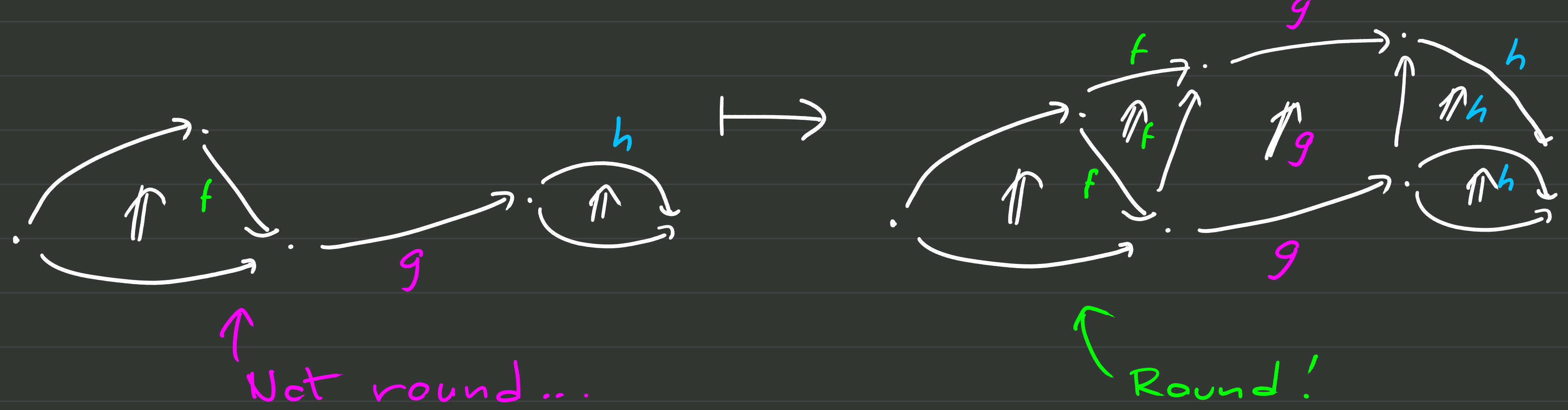
Equivalences in diagrammatic sets

Working with pasting diagrams in a diagrammatic set "feels like" being in a strict ω-category, except only round diagrams can appear as boundaries of cells.

However, there are "weak units" — degenerate pasting diagrams produced by collapses of rdcpxs ...

Equivalences in diagrammatic sets

"Padding" a non-round diagram with units:



Equivalences in diagrammatic sets

In strict n-categories, there is a notion of **weakly invertible cell**, generalising

- isomorphisms in a category,
- equivalences in a 2-category,
- ...

Equivalences in diagrammatic sets

$x \xrightarrow{e} y$ is **weakly invertible** if

$$\exists \quad y \xrightarrow{e^*} x , \quad
\begin{tikzcd}
x \arrow[rr, bend left, "x"] \arrow[dr, "e"'] & & x \\
& y \arrow[ur, "e^*"'] \arrow[u, phantom, "\Uparrow z"] &
\end{tikzcd}
, \quad
\begin{tikzcd}
& x \arrow[dr, "e"] & \\
y \arrow[ur, "e^*"] \arrow[rr, bend right, "y"'] & \arrow[u, phantom, "\Uparrow h"] & y
\end{tikzcd}$$

s.t. z, h are **weakly invertible**

Equivalences in diagrammatic sets

$x \xrightarrow{e} y$ is **weakly invertible** if

$$\exists \quad y \xrightarrow{e^*} x \ , \quad
\begin{tikzcd}
 & y \arrow[dr, "e^*"] & \\
x \arrow[ur, "e"] \arrow[rr, bend left=60, "x"] & \arrow[u, phantom, "\Uparrow z"] & x
\end{tikzcd}
\ , \quad
\begin{tikzcd}
 & x \arrow[dr, "e"] & \\
y \arrow[ur, "e^*"] \arrow[rr, "y"'] & \arrow[u, phantom, "\Uparrow h"] & y
\end{tikzcd}$$

s.t. z, h are **weakly invertible**

OBSERVATION:

This definition only needs **pasting** & **units**!

Equivalences in diagrammatic sets

Instantiating the definition for round diagrams in a diagrammatic set determines a class

ROUND DIAGRAMS

$$\mathrm{Dgn}\, X \subset \mathrm{Eqv}\, X \subset \mathrm{Rd}\, X$$

DEGENERATE ROUND DIAGRAMS

the equivalences in X

Equivalences in diagrammatic sets

Given parallel round diagrams $u, v \in \mathrm{Rd}\, X_n$

$$u \simeq v$$

iff

$$\exists\, h \in \mathrm{Eqv}\, X_{n+1}, \quad \partial^- h = u, \quad \partial^+ h = v.$$

This is an equivalence relation on $\mathrm{Rd}\, X$ ...

LAYER 4

Composition

$\rightsquigarrow$

Diagrammatic (∞, n)-categories

C.C., A.H., *Model Structures for Diagrammatic (∞, n)-Categories*

will be on the arxiv next week!

Diagrammatic (∞, n)-categories

Definition

An (∞,∞)-category is a diagrammatic set X such that

$\forall$ round diagrams u in X

$\exists$ a cell $\langle u \rangle$ in X

such that $u \simeq \langle u \rangle$.

$\langle u \rangle \equiv$ WEAK COMPOSITE OF u

Diagrammatic (∞, n) - categories

$$u \equiv \begin{tikzcd} \overset{x}{\bullet} \arrow[r, "f"] & \overset{y}{\bullet} \arrow[r, "g"] & \overset{z}{\bullet} \arrow[r, "h"] & \overset{w}{\bullet} \end{tikzcd}$$

$$\langle u \rangle \equiv \begin{tikzcd} \overset{x}{\bullet} \arrow[r, "hgf"] & \bullet\, w \end{tikzcd}$$

$$u \simeq \langle u \rangle \equiv \begin{tikzcd} \bullet \arrow[rrr, bend left, "hgf"] \arrow[r, "f"'] & \bullet \arrow[r, "g"'] \arrow[r, phantom, "\Uparrow c", shift left=4] & \bullet \arrow[r, "h"'] & \bullet \end{tikzcd} \quad , \quad c \in \mathrm{Equ}\, X$$

Diagrammatic (∞, n)-categories

Definition

Let $n \in \mathbb{N}$. An (∞,∞)-category X is an (∞, n)-category if every cell of dimension $> n$ in X is an equivalence.

Diagrammatic (∞, n)-categories

Definition

Let X, Y be (∞, ∞)-categories.

A *functor* $f : X \longrightarrow Y$ is a morphism of

diagrammatic sets

WEAK COMPOSITES ARE AUTOMATICALLY PRESERVED!

Diagrammatic (∞, n)-categories

Definition

A functor $f : X \longrightarrow Y$ is an *ω-equivalence* if it is essentially surjective on cells of each dimension

UP TO $\simeq$

Diagrammatic (∞, n)-categories

Theorem

For each $n \in \mathbb{N} \cup \{\infty\}$,

There exists a model structure on $\odot \underline{Set}$ whose

- cofibrations $\equiv$ monomorphisms
- fibrant objects $\equiv$ (∞, n)-categories
- weak equ btw. fibrants $\equiv$ ω-equivalences

Diagrammatic (∞, n)-categories

Theorem (Homotopy Hypothesis)

The $(\infty, 0)$-model structure coincides with the "test category" model structure

(so it is Quillen-equivalent to the classical model structure on $s\underline{Set}$ — in 2 ways!)

Future work

- Comparison with complicial model
- Semi-strictification
- Connection to algebraic models

Thanks for listening!

www.ioc.ee/~naman