PASTING DIAGRAMS BEYOND ACYCLICITY

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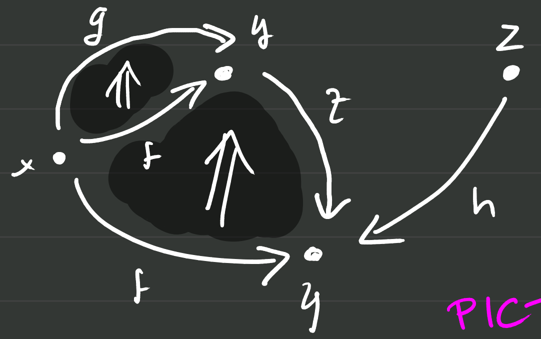
CT 2024

A book on the

Combinatorics of Higher-Categorical Diagrams

arXiv: 2404.07273

DIAGRAMS IN n-CATEGORIES



PICTURE

A $\boxed{\text{cell complex}}$

CONCEPT

+

A $\boxed{\text{direction}}$ on its cells

+

A $\boxed{\text{labelling}}$ defining a functor

MATHEMATICAL STRUCTURE

Parity complexes, Pasting schemes, Directed complexes ...

STRUCTURES FOR DIAGRAMS

UNDERLYING DATA	Combinatorial :	Street, Johnson, Steiner 1993, Forest
	Topological :	Power, Kapranov-Voevodsky
	Algebraic :	Steiner 2004
CHARACTERISATION OF "WELL-FORMED" STRUCTURES	Synthetic :	Steiner 1993 (partially)
	Analytic :	Everyone else

PASTING DIAGRAMS $\subset$ DIAGRAMS *

A *pasting diagram* admits a *composite*, a single cell obtained from the composition of all cells in the diagram

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

PASTING

$$\begin{tikzcd} \bullet \arrow[dr] & & \bullet \arrow[dl, shift left] \arrow[dl, shift right] \\ & \bullet & \end{tikzcd}$$

NON-PASTING

* my own choice of terminology!

ACYCLICITY CONDITIONS

Essentially all combinatorial formalisms impose an acyclicity condition on well-formed structures which bars, at least,

from appearing in a diagram

MOTIVATIONS FOR ACYCLICITY

VALIDITY

1. Exclude a large class of unwanted "non-examples"
2. Ensure that subsets of cells form an n-category
3. Ensure that this n-category is freely generated (a polygraph/computad)

DOWNSIDES OF ACYCLICITY

① It excludes also plenty of good examples:

NON-PASTING IN DIM. 1

$$\begin{tikzcd} x \arrow[r, bend left, "f"] & y \arrow[l, bend left, "g"] \end{tikzcd}$$

PASTING IN DIM. 3

$$\begin{tikzcd}
 & \mathcal{C} \arrow[r, bend left, "R"] \arrow[d, "R"] \arrow[dl, phantom, "\Uparrow \mathrm{Id}_R"] & \mathcal{D} \arrow[dl, phantom, "\Uparrow \mathrm{Id}_R"] \\
\mathcal{C} \arrow[ur, "\mathrm{Id}_{\mathcal{C}}"] \arrow[r, "R"'] & \mathcal{D} \arrow[ur, "\mathrm{Id}_{\mathcal{D}}"'] &
\end{tikzcd}
\quad \Rrightarrow \quad
\begin{tikzcd}
 & \mathcal{C} \arrow[dr, "R"] & \\
\mathcal{C} \arrow[ur, "\mathrm{Id}_{\mathcal{C}}"] \arrow[dr, "R"'] \arrow[r, phantom, "\Uparrow \varepsilon"] & {} & \mathcal{D} \arrow[l, phantom, "\Uparrow \eta"'] \\
 & \mathcal{D} \arrow[uu, "L"] \arrow[ur, "\mathrm{Id}_{\mathcal{D}}"'] &
\end{tikzcd}$$

DOWNSIDES OF ACYCLICITY

② It is an unstable global property:

- STRONGER acyclicity properties are not stable under duals
- WEAKER acyclicity properties are not stable under Gray products, pasting, joins

ACYCLICITY IS RARE

Forthcoming work by G. Laplante-Anfossi, A. Medina-Mardones, A. Padrol:

> The density of acyclic directions on the n-simplex among all well-formed directions tends to 0 as $n \to \infty$.

Street's oriented simplices are the others ...

LEARNING TO LIVE WITH CYCLES

1 Replace acyclicity with regularity (as in regular cell complex):

$$\boxed{\text{The source \& target of an } n\text{-cell are round (closed topological } (n-1)\text{-balls)}}$$

Corresponds to the combinatorial condition:

$$\forall k < n \qquad \partial_k^+ U \cap \partial_k^- U = \partial_{k-1}^+ U \cup \partial_{k-1}^- U$$

LEARNING TO LIVE WITH CYCLES

Regularity is

- Local
- Stable under all sorts of constructions
- Ensures that a cell complex can be reconstructed from its face poset

LEARNING TO LIVE WITH CYCLES

2. Given a directed graph $\mathcal{G}$,

Linear subgraphs form a category iff $\mathcal{G}$ is acyclic, but

Paths in $\mathcal{G}$ always form a category ...

$\Rightarrow$ Move from subsets to more general morphisms

THE CATEGORY ogPos

- Objects are oriented graded posets, which are graded posets together with an orientation: $\Delta x = \Delta^+ x + \Delta^- x$

FACES OF x OUTPUT FACES INPUT FACES

- Morphisms $f : P \longrightarrow Q$ are functions inducing bijections $\Delta^\alpha x \xrightarrow{\sim} \Delta^\alpha f(x)$ $\forall x, \alpha \in \{+, -\}$.

MOLECULES

Inductive subclass of o.g. posets corresponding to "shapes of pasting diagrams":

① The point $\bullet$ is a molecule,

② If U, V are molecules, pasting along an iso $\partial_k^+ U \xrightarrow{\sim} \partial_k^- V$ produces a molecule $U \#_k V$,

③ If U, V are round, n-dim molecules, gluing along $\partial_{n-1} U \xrightarrow{\sim} \partial_{n-1} V$ and adding an $(n+1)$-dim cell produces a molecule $U \Rightarrow V$

PUSHOUTS IN ogPos

PROPERTIES OF MOLECULES

- Molecules are rigid (no non-trivial automorphisms)
- Molecules are stable under boundary, pasting, suspension, Gray products, joins, duals ...
- Geometric realisations of molecules are wedges of balls (in particular contractible)
- Iso classes of molecules form a strict ω-category with $\#_k$ as k-composition

MOLECULES OVER P

Let P be an o.g. poset

$\mathrm{Mol}/_P$:= iso-classes in $\mathrm{ogPos}/_P$ of morphisms $f : U \to P$, U a molecule

Then $\mathrm{Mol}/_P$ has a structure of strict ω-category with "fibred pasting" as composition.

REGULAR DIRECTED COMPLEXES

LOWER SET

P o.g. poset such that $\forall x \in P$, $\mathrm{cl}\{x\}$ is a molecule.

The underlying poset of a regular directed complex is the face poset of a regular CW complex!

Then $\mathrm{Mol}/_P$ is generated by $\{[\mathrm{cl}\{x\} \hookrightarrow P] \mid x \in P\}$

LEARNING TO LIVE WITH CYCLES

③ In general, $\mathrm{Mol}/_{P}$ is not freely generated as an ω-category

↳ Acyclicity does play a role!

(Q : Do you need freeness, and why?)

FRAME-ACYCLIC MOLECULES

A milder, technical acyclicity condition suffices for free generation...

$\boxed{\text{This ALWAYS HOLDS in } \dim \leq 3\,!}$

STRICT BOUND

If $\dim P \leq 3$, then $\mathrm{Mol}/_P$ is a polygraph.

EMBRACE THE CYCLES

... If you absolutely need to get computads for strict ω-categories, up to dim. 3;

... If not, in all dimensions!

THANKS FOR LISTENING!

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