Units without degeneracy, from polycategories to sequent calculi

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Trouble with units in topology and logic

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No proof nets for MLL with units

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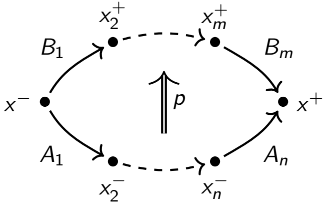
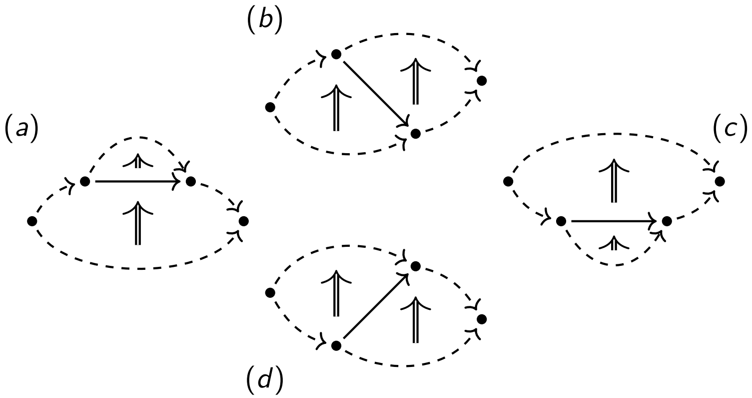
- 2-cells $p, q, \ldots : (A_1, \ldots, A_n) \to (B_1, \ldots, B_m)$

Topology: disks; *Logic*: sequents



Composition (cut)



Composition (cut)

$$\frac{\Gamma_1 \vdash \Delta_1, A \qquad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \mathrm{CUT}_b$$

$$\frac{\Gamma \vdash \Delta_1, A, \Delta_2 \qquad A \vdash \Delta}{\Gamma \vdash \Delta_1, \Delta, \Delta_2}\ \mathrm{CUT}_a \qquad \frac{\Gamma \vdash A \qquad \Gamma_1, A, \Gamma_2 \vdash \Delta}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta}\ \mathrm{CUT}_c$$

$$\frac{\Gamma_2 \vdash A, \Delta_2 \qquad \Gamma_1, A \vdash \Delta_1}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \mathrm{CUT}_d$$

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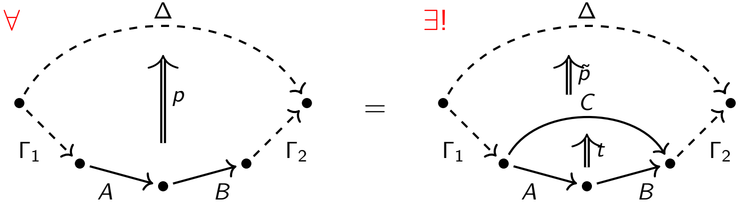
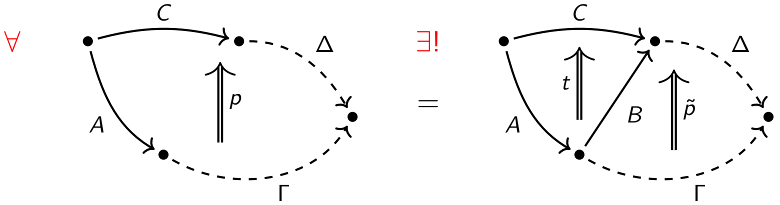


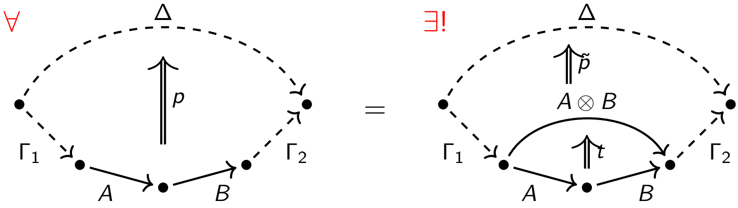
Divisible 2-cells

A 2-cell $t : (A, B) \to (C)$ is **divisible at** ∂_2^- if



Divisible 2-cells produce rules of sequent calculus

$t : (A, B) \to (A \otimes B)$ divisible at ∂_1^+:



$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, A \otimes B, \Gamma_2 \vdash \Delta} \otimes_L$$

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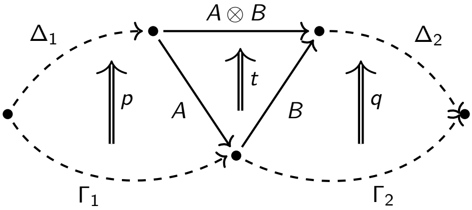

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Units: the usual approach

2-cells $(A_1, \ldots, A_n) \to (A)$, with $n \geq 2$, divisible at ∂_1^+, model **composition of paths** in topology, and *n*-**ary tensors** (or conjunctions) in logic

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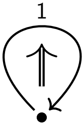
Units/constant paths (in Cockett-Seely and Hermida)
$\rightsquigarrow$ divisible 2-cells with a degenerate boundary (0-ary tensors/pars)



Coherence via universality

Multicategory

A polycategory where all 2-cells have a single output.

($\rightsquigarrow$ intuitionistic sequent calculi)

Representable multicategory

For all composable $(A_1, \ldots, A_n)$, $n \geq 0$, there exists an "n-ary tensor" 2-cell $(A_1, \ldots, A_n) \to (\otimes_{i=1}^{n} A_i)$ divisible at ∂_1^+.

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Hermida, 2000

Monoidal categories and strong monoidal functors are equivalent to representable multicategories (with a choice of divisible 2-cells) and morphisms that preserve divisibility at ∂_1^+.

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Linearly distributive categories and strong linear functors are equivalent to representable polycategories (with a choice of divisible 2-cells) and morphisms that preserve divisibility at ∂_1^+ and ∂_1^-.

So, all's good up to dimension 2...

But:

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A solution: regularity

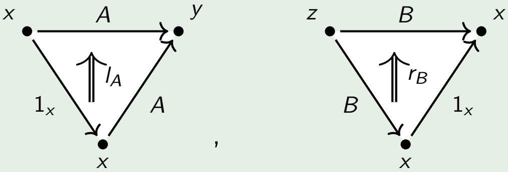
Input and output boundaries of 2-cells are 1-dimensional (in general: k-boundaries of n-cells are k-dimensional)

We need a new definition for units

Idea: Saavedra unit (J. Kock, 2006), reformulated

Tensor unit $1_x : x \to x$

For all $A : x \to y$, $B : z \to x$, there exist



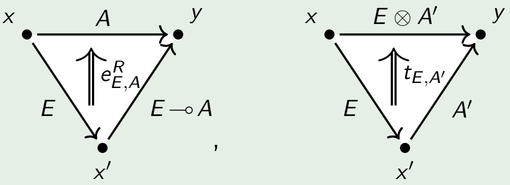
respectively divisible at ∂_1^+ and ∂_2^-, and at ∂_1^+ and ∂_1^-.

Induces the correct coherent structure (triangle equations, etc)

But we can do better

Tensor left divisible 1-cell $E : x \to x'$

For all $A : x \to y$, $A' : x' \to y$, there exist



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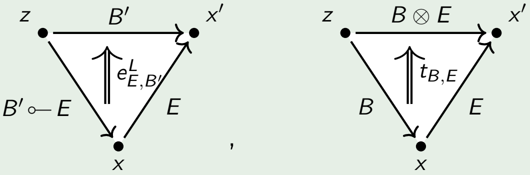

divisible both at ∂_1^+ and ∂_1^-.

Tensor divisible 1-cell $E : x \to x'$

Tensor right and left divisible 1-cell.

From divisible cells to units

Theorem

The following are equivalent in a regular poly-bicategory:

- for all 0-cells x, there exists a tensor unit $1_x : x \to x$;
- for all 0-cells x, there exist a 0-cell $\overline{x}$ and a tensor divisible 1-cell $e : x \to \overline{x}$;
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Representability: existence of enough divisible 2-cells and 1-cells

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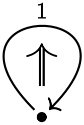
Scales to higher dimensions:

- A.H., *A combinatorial-topological shape category for polygraphs*. (Later this year)

An observation on the sequent calculus side

Tensor units as 0-ary tensors:



$\rightsquigarrow$ introduction of units is a "divisibility property" rule

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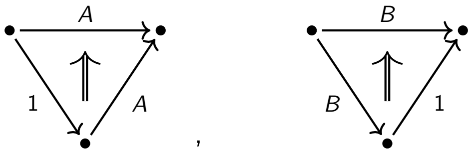

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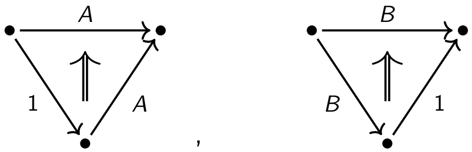

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This difference is not captured by the induced structure

(monoidal categories, etc)

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What does the number of "residual units" count?

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Thank you for your attention.