

THE HOMOTOPY POSETS OF A CATEGORY

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(i) Po(m)set Seminar

17. 11. 2023

Reference:

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Obstructions to Compositionality

ACT 2023

arXiv: 2307.14461

Let \mathcal{C} be a category.

Let $f: x \rightarrow y$ be a morphism in \mathcal{C} .

We can answer the following question:

Is f an isomorphism?

with YES or NO.

Can we give a finer answer?

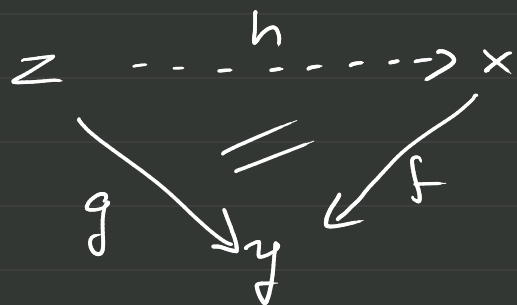
Can we say how far f is from being an isomorphism? Or what stops f from being an isomorphism?

We know that

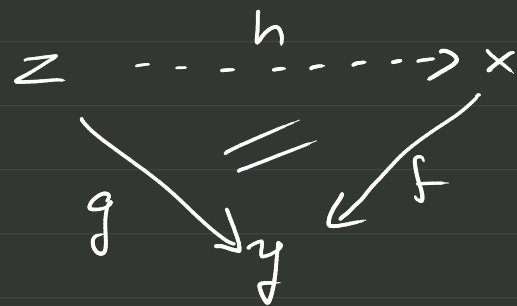
f is ISO $\iff f$ is SPLIT EPI
& MONO

Let's consider SPLIT EPI separately.

$f: x \rightarrow y$ is split epi iff
 $\forall g: z \rightarrow y \exists h: z \rightarrow x$ s.t.



$f: x \rightarrow y$ is split epi iff
 $\forall g: z \rightarrow y \quad \exists h: z \rightarrow x$ s.t.



Every $g: z \rightarrow y$ s.t.
a factorisation $g = hf$ does
not exist can be seen as an
obstruction to f being split epi.

$$f: x \rightarrow y \text{ is } \underset{\text{ISO}}{\text{ISO}} \iff f \text{ is } \underset{\text{in } \mathcal{C}/\mathcal{D}}{\text{TERMINAL}}$$

Through this equivalence:

$$\text{ISO} = \text{SPLIT EPI} + \text{MONO}$$



$$\text{TERMINAL} = \text{WEAK TERMINAL} + \text{SUBTERMINAL}$$

Our contribution:

We will define functors $\mathcal{C} \rightarrow \underline{\text{Pos}}$.

$$\begin{array}{l} x \mapsto \pi_0(\mathcal{C}/x) \\ x \mapsto \pi_1(\mathcal{C}/x) \end{array} \quad \begin{array}{l} \text{valued in} \\ \text{POINTED POSETS} \end{array}$$

s.t. $\pi_0(\mathcal{C}/x) \simeq \{*\}$ iff x is weak
terminal

$\pi_1(\mathcal{C}/x) \simeq \{*\}$ iff x is subterminal

both vanish iff x is terminal

In particular, given $f: x \rightarrow y$,

$$\pi_0\left(\left(\frac{e}{y}\right)/f\right) \cong \{*\} \iff f \text{ is split epi}$$

$$\pi_1\left(\left(\frac{e}{y}\right)/f\right) \cong \{*\} \iff f \text{ is mono}$$

$$\text{both vanish} \iff f \text{ is iso}$$

STEP 1

The inclusion $\underline{\text{Pos}} \hookrightarrow \underline{\text{Cat}}$
has a left adjoint

$$\|-\| : \underline{\text{Cat}} \longrightarrow \underline{\text{Pos}},$$

the poset reflection:

- elements of $\|e\|$ are equivalence classes of objects of e , $\|x\| = \|y\|$ iff $\exists x \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} y$
- $\|x\| \leq \|y\|$ iff $\exists x \longrightarrow y$

Given (e, x) , we take the slice domain fibration

$$\begin{array}{ccc} e/x & \dashv\dashv & \|e/x\| \\ \downarrow \text{dom} & \dashrightarrow & \downarrow \| \text{dom} \| \\ e & & \|e\| \end{array}$$

and apply the poset reflection functor

STEP 2

In Pos, we take the pushout

$$\begin{array}{ccc} \|\mathcal{E}/x\| & \xrightarrow{!} & 1 \\ \text{\scriptsize \|\downarrow\|} \downarrow & & \downarrow [x] \\ \|\mathcal{E}\| & \longrightarrow & \pi_0(\mathcal{E}/x) \end{array}$$

The pair of $(\pi_0(\mathcal{E}/x), [x])$
determines a pointed poset.

Explicitly, elements of $\pi_0(\mathcal{L}/x)$ are either

- $[x]$, or
- $\|y\|$ s.t. $\nexists y \rightarrow x$

and the partial order is

• $[x] \leq [x]$ trivially,

• $[x] \leq \|y\|$ iff $\exists z \begin{matrix} \swarrow \\ x \end{matrix} \begin{matrix} z \\ \searrow \\ y \end{matrix}$,

• never $\|y\| \leq [x]$,

• $\|y\| \leq \|z\|$ iff $\exists y \rightarrow z$.

Why is it called π_0 ?

Proposition

Suppose \mathcal{G} is a groupoid, $x \in \text{Ob}(\mathcal{G})$.

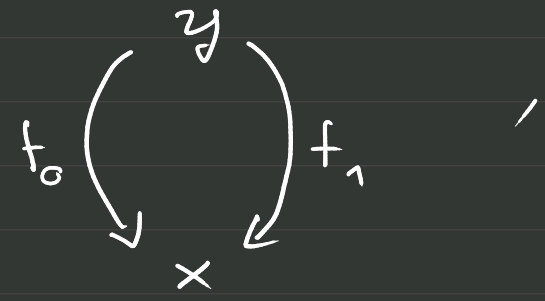
Then

- ① $\pi_0(\mathcal{G}/x)$ is a discrete poset,
i.e. a "set",
- ② isomorphic to the set $\pi_0(\mathcal{G})$ of
connected components of \mathcal{G} ,
- ③ pointed with the connected
component of x .

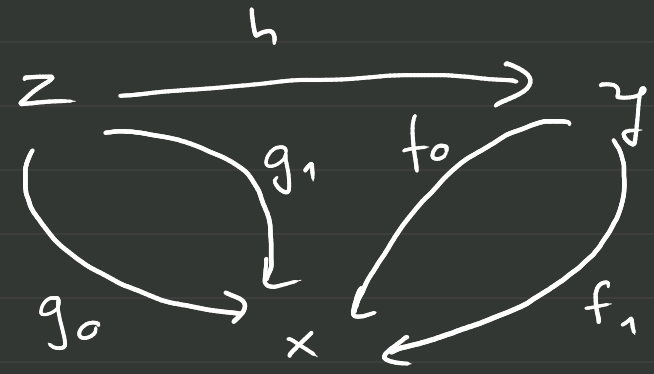
STEP 3

Def The category $\text{Par}(e/x)$ of parallel arrows over x has

- objects pairs



- morphisms



s.t. $h; f_0 = g_0$,
 $h; f_1 = g_1$

Proposition TFAE:

a) x is subterminal in \mathcal{C}

b) $\begin{array}{c} x \\ \downarrow \text{id}_x \quad \uparrow \text{id}_x \\ x \end{array}$ is terminal in $\text{Par}(\mathcal{C}/x)$

c) $\begin{array}{c} x \\ \downarrow \text{id}_x \quad \uparrow \text{id}_x \\ x \end{array}$ is **weak** terminal in $\text{Par}(\mathcal{C}/x)$

Def We let

$$\left(\pi_1(e/x), [x] \right)$$

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$$\left(\pi_0 \left(\text{Par}(e/x) / \begin{array}{c} \text{id}_x \\ \text{id}_x \end{array} \right), \left[\begin{array}{c} \text{id}_x \\ \text{id}_x \end{array} \right] \right)$$

Compare with: $\pi_1(X, x) :=$
 $\pi_0(\Omega(X, x), c_x)$

Explicitly, elements of $\pi_1(\mathcal{C}/x)$ are either

- $[x]$, or

- $\|t_0, t_1\|$ for a pair $t_0 \begin{pmatrix} y \\ \vdots \\ x \end{pmatrix} t_1$

with the partial order

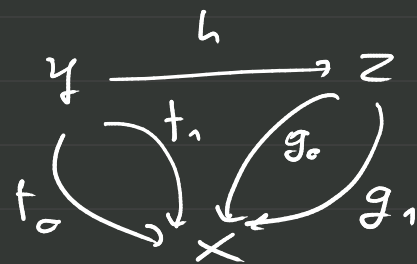
- $[x] \leq [x]$ trivially,

- $[x] \leq \|t_0, t_1\|$ iff $\exists z \xrightarrow{h} y$

s.t. $h; f_0 = h; f_1$,

- never $\|t_0, t_1\| \leq [x]$,

- $\|t_0, t_1\| \leq \|g_0, g_1\|$ iff \exists



Why is it called π_1 ?

Proposition

Suppose \mathcal{G} is a groupoid, $x \in \text{Ob}(\mathcal{G})$.

Then

① $\pi_1(\mathcal{G}/x)$ is a discrete poset,
i.e. a "set"

② isomorphic to the underlying
pointed set of the automorphism
group $\text{Aut}(x) \equiv \pi_1(\mathcal{G}, x)$.

Explicitly, the isomorphism

$$\pi_1(\mathcal{Y}/x) \longrightarrow \text{Aut}(x)$$

is defined by

$$[x] \longmapsto \begin{array}{c} \text{id}_x \\ \curvearrowright \\ x \end{array}$$

$$\left\| \begin{array}{c} y \\ \begin{array}{c} \curvearrowright \\ f \end{array} \\ x \\ \begin{array}{c} \curvearrowright \\ g \end{array} \end{array} \right\| \longmapsto \begin{array}{c} g^{-1} \\ \curvearrowright \\ x \\ \curvearrowright \\ f \end{array}$$

"A group is an
"undirected monoid" —
we need a fundamental
monoid instead of a
fundamental group"

← TYPICALLY,
IN DIRECTED
ALGEBRAIC
TOPOLOGY

HERE →

A pointed set is an
"undirected pointed poset";
the group structure is
'incidental' and has no
directed counterpart.

Example

Let $f: X \rightarrow Y$ be a function.

We will compute

$$\pi_0\left(\left(\frac{\text{Set}}{Y}\right)/f\right) \quad (\text{"obstructions to surjectiveness"})$$

$$\pi_1\left(\left(\frac{\text{Set}}{Y}\right)/f\right) \quad (\text{"obstructions to injectiveness"})$$

Claim

$$\begin{array}{ccc} \mathcal{P}(f(X)) & & \left\| \left(\frac{\text{Set}}{y} \right) / f \right\| \\ \downarrow & \approx & \downarrow \text{down} \\ \mathcal{P}(y) & & \left\| \frac{\text{Set}}{y} \right\| \end{array}$$

$$S \subseteq y \quad \mapsto \quad \left\| \begin{array}{c} S \\ \downarrow \\ y \end{array} \right\|$$

Then in $\pi_0\left(\frac{\text{Set}/y}{f}\right)$ all $S \subseteq f(x)$ are identified with $[f]$. The result is isomorphic to the subposet of $\mathcal{P}(y)$ on

$$\left\{ S \subseteq y \mid \exists x \in S \setminus f(x) \right\}.$$

Here

- $[f] \equiv \emptyset$ is the least element,
- The atoms (minimal elements above \emptyset) are the singletons $\{y\}$, $y \in y \setminus f(x)$

Claim

$$\text{Let } X \times_f X := \{ (x_0, x_1) \mid f x_0 = f x_1 \},$$

$$\Delta X := \{ (x, x) \mid x \in X \} \subseteq X \times_f X.$$

$$\begin{array}{ccc} \Phi(\Delta X) & \parallel \text{Par}(\frac{\text{Set}/y}{f}) / (id_f, id_f) \parallel & \\ \downarrow & \downarrow \parallel \text{dom} \parallel & \\ \Phi(X \times_f X) & \approx \parallel \text{Par}(\frac{\text{Set}/y}{f}) \parallel & \end{array}$$



Then in $\pi_1(\underline{\text{Set}}/y)/f$ any $S \subseteq \Delta X$
are identified with $[f]$.

The result is isomorphic to the subposet
of $\Phi(X \times_f X)$ on

$$\left\{ S \subseteq X \times_f X \mid \exists (x_0, x_1) \in S \text{ s.t. } x_0 \neq x_1 \right\}.$$

Here

- $[f] \equiv \emptyset$ is the least element,
- the atoms are the singletons
 $\{(x_0, x_1)\}$ with $x_0 \neq x_1$, $f x_0 = f x_1$.

FUNCTORIALITY

For $i \in \{0, 1\}$,

$$x \mapsto \pi_i(\mathcal{L}/x)$$

extends to a functor

$$\pi: (\mathcal{L}/-) : \mathcal{L} \longrightarrow \underline{\text{Pos}}.$$

i.e. for all $f: x \rightarrow y$, we have

basepoint-preserving, order-preserving maps

$$f_*: \pi(\mathcal{L}/x) \longrightarrow \pi(\mathcal{L}/y)$$

Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor.

Then we have a natural transformation

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \pi_i(\mathcal{C}/-) \searrow & \xRightarrow{\pi_i F} & \swarrow \pi_i(\mathcal{D}/-) \\ & \underline{\text{Pos.}} & \end{array}$$

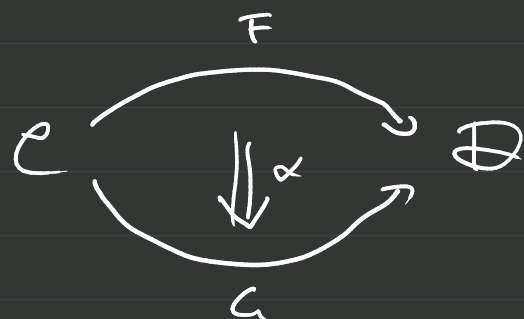
This is compatible with identities & composition.

In brief: π_i defines a functor

$$\underline{\text{Cat}} \longrightarrow \underline{\underline{\text{Cat}}}/\underline{\underline{\text{Pos.}}}^{\text{lex}}$$

FUNCTORIALITY, take 2

Consider a natural transformation



Then there are functors $\mathcal{C} \rightarrow \underline{\text{Pos}}$ defined on objects by

$$x \longmapsto \pi_i \left(\frac{(\mathcal{D}/G_x)}{\alpha_x} \right)$$

$$x \longmapsto \pi: \left((\mathbb{D}/\mathfrak{a}_x) / \mathfrak{a}_x \right)$$

Recall that the elements of these posets are "obstructions to $\alpha_x: F_x \rightarrow G_x$ being split epi/mono".



Naturality of α induces a functorial "flow" in the invariants associated to its components

EXISTENCE OF JOINS

Proposition

Let \mathcal{C} be a category, κ a cardinal,
 $x \in \text{ob}(\mathcal{C})$.

If \mathcal{C} has κ -small coproducts,
then $\pi_0(\mathcal{C}/x)$ and $\pi_1(\mathcal{C}/x)$ have
 κ -small joins.

(Meets are w.i.p...)

MONOIDAL STRUCTURE

In general, there is **no monoid structure** on $\pi: (\mathcal{C}/x)$.

Under certain conditions, however, a monoidal category structure $(\mathcal{C}, \otimes, i)$ together with an internal monoid structure $(m: x \otimes x \rightarrow x, e: i \rightarrow x)$ can induce it.

FACT 1

The poset reflection $\|- \|-: \underline{\text{Cat}} \rightarrow \underline{\text{Pos}}$
lifts to a functor $\underline{\text{MonCat}} \rightarrow \underline{\text{MonPos}}$,
i.e. a monoidal category reflects onto
a monoidal poset.

FACT 2

A monoid object $(m: x \otimes x \rightarrow x, e: i \rightarrow x)$
determines a monoidal structure (\otimes_m, e)
on $\mathcal{E}/_x$ and on $\text{Par}(\mathcal{E}/_x)$; the
domain fibration is monoidal.

\Rightarrow Given a monoid (x, m, e) in $(\mathcal{C}, \otimes, i)$ we get an order-preserving homomorphism

$$\|(e/x, \otimes_m, e)\|$$

$$\downarrow \|\text{down}\|$$

$$\|(e, \otimes, i)\|$$

However, in general, the pushout
defining $\pi_0(\mathcal{C}/x)$ in Pos does
NOT lift to a pushout (**cokernel**)
in MonPos ...

↳ This is only true if
the equivalence relation
generated by the image of
 $\| \text{down} \|$ is a **congruence** for
multiplication in $\| \mathcal{C} \|$

Lemma TFAE:

• $\pi_0(\mathcal{C}/x)$ admits a monoidal structure,
s.t. $\|\mathcal{C}\| \twoheadrightarrow \pi_0(\mathcal{C}/x)$ is the
cokernel of $\|\text{dom}\|$;

• for all y, z s.t. $\|y\| \leq \|x\|$,
either $\|z\| \leq \|x\|$ or
 $\|z\| = \|y\| \cdot \|z\| = \|z\| \cdot \|y\|$.

(In particular: $\forall y$ either $\|y\| \leq \|x\|$ or
 $\|y\| = \|x\| \cdot \|y\| = \|y\| \cdot \|x\|$)

There is a similar story for the π_1 ,
 starting from an order-preserving
 homomorphism


$$\left\| \text{Par}(e/x) / \downarrow_x \left(\begin{array}{c} x \\ \downarrow_x \\ x \end{array} \right) \downarrow_x \right\|$$

$$\downarrow \|\downarrow\|$$

$$\|\text{Par}(e/x)\|$$

...

But it turns out there's a simpler "sufficient,
 almost necessary" condition ...

Def $x \in \text{Ob}(\mathcal{C})$ is a weak zero object
if $\forall y \in \text{Ob}(\mathcal{C}) \exists$ 

Proposition (x, m, e) monoid in $(\mathcal{C}, \otimes, i)$

① If $e: i \rightarrow x$ is a weak zero object
in \mathcal{C}/x , then $\pi_1(\mathcal{C}/x)$ has a monoidal
structure, s.t. $\| \text{Par}(\mathcal{C}/x) \| \rightarrow \pi_1(\mathcal{C}/x)$
is the cokernel of $\| \text{Idem} \|$.

② If $\exists s: i \rightarrow x$ s.t. $s \neq e$, then the
converse implication also holds.

Theorem (x, m, e) monoid in $(\mathcal{C}, \otimes, i)$

① If $e: i \rightarrow x$ is a weak zero object in \mathcal{C}/x ,

then a) $\|e\| \rightarrow \pi_0(\mathcal{C}/x)$ is an

isomorphism,

b) $\|\text{Par}(\mathcal{C}/x)\| \rightarrow \pi_1(\mathcal{C}/x)$ is

a cokernel;

② If $\exists s: i \rightarrow x$, $s \neq e$, the

converse implication also holds.

Example

If (x, m, e) is a monoid in a monoidal groupoid $(\mathcal{C}, \otimes, i)$, then e is always a weak zero object in \mathcal{C}/x .

So we get monoid structures on $\pi_0(\mathcal{C}/x)$, $\pi_1(\mathcal{C}/x)$

Example

If $(\mathcal{C}, \oplus, 0)$ is semicocartesian, i.e.

0 is initial, then

① 0 has a "trivial" monoid structure

$$0 \oplus 0 \xrightarrow{\sim} 0, \quad 0 \xrightarrow{\text{id}_0} 0$$

② id_0 is a zero object in $\mathcal{C}/0$

So the conditions are satisfied for

$$\pi_0(\mathcal{C}/0), \quad \pi_1(\mathcal{C}/0)$$

Example: $(\underline{\text{Ring}}, \otimes, \mathbb{Z})$

In general, given a monoidal category $(\mathcal{C}, \otimes, i)$, the slice i/e is always cocartesian monoidal with unit $i \xrightarrow{\text{id}_i} i$

$\Rightarrow \pi_0\left(\frac{(i/e)}{\text{id}_i}\right), \pi_1\left(\frac{(i/e)}{\text{id}_i}\right)$
 have monoidal structures

