THE HOMOTOPY POSETS OF A CATEGORY

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Reference:

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Obstructions to Compositionality

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Let $\mathcal{C}$ be a category.
Let $f: x \to y$ be a morphism in $\mathcal{C}$.

We can answer the following question:

Is f an isomorphism?

with YES or NO.

Can we give a finer answer?
Can we say how far f is from being an isomorphism? Or what stops f from being an isomorphism?

We know that

f is ISO $\iff$ f is SPLIT EPI
& MONO

Let's consider SPLIT EPI separately.

$f : x \to y$ is split epi iff

$\forall g : z \to y \quad \exists h : z \to x \quad$ s.t.

$$\begin{tikzcd}
z \arrow[rr, dashed, "h"] \arrow[dr, "g"'] & & x \arrow[dl, "f"] \\
& y \arrow[u, phantom, "="] &
\end{tikzcd}$$

$f : x \longrightarrow y$ is split epi iff

$\forall\, g : z \longrightarrow y \quad \exists h : z \longrightarrow x$ s.t.

$$\begin{tikzcd} z \arrow[rr, dashed, "h"] \arrow[dr, "g"'] & \arrow[d, phantom, "="] & x \arrow[dl, "f"] \\ & y & \end{tikzcd}$$

Every $g : z \longrightarrow y$ s.t. a factorisation $g = h ; f$ does not exist can be seen as an obstruction to f being split epi.

$f : x \longrightarrow y$ is ISO $\iff$ f is TERMINAL in $\mathcal{C}/y$

Through this equivalence:

$$\boxed{\text{ISO} \quad = \quad \text{SPLIT EPI} + \text{MONO}}$$

$$\downarrow$$

$$\boxed{\text{TERMINAL} \quad = \quad \text{WEAK TERMINAL} + \text{SUBTERMINAL}}$$

Our contribution:

We will define functors $\mathcal{C} \longrightarrow \underline{\mathrm{Pos}}_\bullet$

$$x \longmapsto \pi_0(\mathcal{C}/_x)$$

valued in

$$x \longmapsto \pi_1(\mathcal{C}/_x)$$

POINTED POSETS

s.t. $\pi_0(\mathcal{C}/_x) \simeq \{*\}$ iff x is weak terminal

$\pi_1(\mathcal{C}/_x) \simeq \{*\}$ iff x is subterminal

both vanish iff x is terminal

In particular, given $f: x \longrightarrow y$,

$\pi_0\left(\left(\mathcal{C}/_y\right)/_f\right) \simeq \{*\} \iff f$ is split epi

$\pi_1\left(\left(\mathcal{C}/_y\right)/_f\right) \simeq \{*\} \iff f$ is mono

both vanish $\iff f$ is iso

$\boxed{\text{STEP 1}}$

The inclusion $\underline{\mathrm{Pos}} \hookrightarrow \underline{\mathrm{Cat}}$ has a left adjoint

$$\| - \| : \underline{\mathrm{Cat}} \longrightarrow \underline{\mathrm{Pos}},$$

the poset reflection:

- elements of $\|\mathcal{C}\|$ are equivalence classes of objects of $\mathcal{C}$, $\|x\| = \|y\|$ iff $\exists\ x \rightleftarrows y$
- $\|x\| \leq \|y\|$ iff $\exists\ x \longrightarrow y$

Given $(\mathcal{C}, x)$, we take the slice domain fibration

$$
\begin{tikzcd}
\mathcal{C}/x \arrow[d, "\mathrm{dom}"] & \overset{\|-\|}{\longmapsto} & \|\mathcal{C}/x\| \arrow[d, "\|\mathrm{dom}\|"] \\
\mathcal{C} & & \|\mathcal{C}\|
\end{tikzcd}
$$

and apply the poset reflection functor

$\boxed{\text{STEP 2}}$

In $\underline{\mathbf{Pos}}$, we take the pushout

$$\begin{tikzcd} \|\mathcal{C}/x\| \arrow[r, "!"] \arrow[d, "\|\mathrm{dom}\|"'] & 1 \arrow[d, "{[x]}"] \\ \|\mathcal{C}\| \arrow[r] & \pi_0(\mathcal{C}/x) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The pair of $(\pi_0(\mathcal{C}/x), [x])$ determines a pointed poset.

Explicitly, elements of $\pi_0(\mathcal{C}/x)$ are either

- $[x]$, or
- $\|y\|$ s.t. $\nexists\ y \longrightarrow x$

and the partial order is

- $[x] \leq [x]$ trivially,
- $[x] \leq \|y\|$ iff $\exists$ $\begin{tikzcd} & z \arrow[dl] \arrow[dr] & \\ x & & y \end{tikzcd}$,
- never $\|y\| \leq [x]$,
- $\|y\| \leq \|z\|$ iff $\exists\ y \longrightarrow z$.

Why is it called π_0?

Proposition

Suppose $\mathcal{G}$ is a groupoid, $x \in Ob(\mathcal{G})$.
Then

1. $\pi_0(\mathcal{G}/x)$ is a discrete poset,
   i.e. a "set",
2. isomorphic to the set $\pi_0(\mathcal{G})$ of connected components of $\mathcal{G}$,
3. pointed with the connected component of x.

STEP 3

Def The category $\mathrm{Par}(\mathcal{C}/x)$ of parallel arrows over x has

- objects pairs

$$\begin{tikzcd} y \arrow[d, bend right, "f_0"'] \arrow[d, bend left, "f_1"] \\ x \end{tikzcd},$$

- morphisms

$$\begin{tikzcd} z \arrow[rr, "h"] \arrow[dr, bend right, "g_0"'] \arrow[dr, bend left, "g_1"] & & y \arrow[dl, bend right, "f_0"'] \arrow[dl, bend left, "f_1"] \\ & x & \end{tikzcd}$$

s.t. $h;f_0 = g_0$,

$h;f_1 = g_1$

Proposition TFAE:

a) x is subterminal in $\mathcal{C}$

b) $\begin{tikzcd} x \arrow[d, bend right, "id_x"'] \arrow[d, bend left, "id_x"] \\ x \end{tikzcd}$ is terminal in $\mathrm{Par}(\mathcal{C}/x)$

c) $\begin{tikzcd} x \arrow[d, bend right, "id_x"'] \arrow[d, bend left, "id_x"] \\ x \end{tikzcd}$ is weak terminal in $\mathrm{Par}(\mathcal{C}/x)$

Def We let

$$\left(\pi_1\left(\mathcal{C}/x\right),\ [x]\right)$$

$$:=$$

$$\left(\pi_0\left(\mathrm{Par}\left(\mathcal{C}/x\right)\Big/ \left(x \overset{\mathrm{id}_x}{\underset{\mathrm{id}_x}{\rightrightarrows}} x\right)\right),\ \left[x \overset{\mathrm{id}_x}{\underset{\mathrm{id}_x}{\rightrightarrows}} x\right]\right)$$

$$\boxed{\text{Compare with: } \begin{array}{l} \pi_1(X, x) := \\ \pi_0\left(\Omega(X, x),\ c_x\right) \end{array}}$$

Explicitly, elements of $\pi_1(\mathcal{C}/x)$ are either

- $[x]$, or
- $\| f_0, f_1 \|$ for a pair $\begin{tikzcd} y \arrow[d, bend right, "f_0"'] \arrow[d, bend left, "f_1"] \arrow[d, phantom, "\neq"] \\ x \end{tikzcd}$

with the partial order

- $[x] \leq [x]$ trivially,
- $[x] \leq \| f_0, f_1 \|$ iff $\exists\, z \xrightarrow{h} y$

  s.t. $h ; f_0 = h ; f_1$,
- never $\| f_0, f_1 \| \leq [x]$,
- $\| f_0, f_1 \| \leq \| g_0, g_1 \|$ iff $\exists$ $\begin{tikzcd} y \arrow[rr, "h"] \arrow[dr, bend right, "f_0"'] \arrow[dr, bend left, "f_1"] & & z \arrow[dl, bend right, "g_0"'] \arrow[dl, bend left, "g_1"] \\ & x & \end{tikzcd}$

Why is it called π_1?

Proposition

Suppose $\mathcal{G}$ is a groupoid, $x \in \mathrm{Ob}(\mathcal{G})$.

Then

① $\pi_1(\mathcal{G}/x)$ is a discrete poset,
i.e. a "set",

② isomorphic to the underlying pointed set of the automorphism group $\mathrm{Aut}(x) \equiv \pi_1(\mathcal{G}, x)$.

Explicitly, the isomorphism

$$\pi_1(\mathcal{G}/x) \longrightarrow \mathrm{Aut}(x)$$

is defined by

$$[x] \longmapsto \begin{tikzcd} x \arrow[loop above, "\mathrm{id}_x"] \end{tikzcd},$$

$$\left\| \begin{tikzcd} y \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] \\ x \end{tikzcd} \right\| \longmapsto \begin{tikzcd} x \arrow[r, bend left, "g^{-1}"] & {} \arrow[l, bend left, "f"] \end{tikzcd}$$

"A group is an "undirected monoid" — we need a fundamental monoid instead of a fundamental group"

TYPICALLY, IN DIRECTED ALGEBRAIC TOPOLOGY

A pointed set is an "undirected pointed poset"; the group structure is 'incidental' and has no directed counterpart.

HERE

$\boxed{\text{Example}}$

Let $f : X \longrightarrow Y$ be a function.

We will compute

$$\pi_0\left(\left(\underline{\mathrm{Set}}/Y\right)/f\right) \quad (\text{"obstructions to surjectiveness"})$$

$$\pi_1\left(\left(\underline{\mathrm{Set}}/Y\right)/f\right) \quad (\text{"obstructions to injectiveness"})$$

$\boxed{\text{Claim}}$

$$\begin{tikzcd} \mathcal{P}(f(X)) \arrow[d, hook] & & \left\| \left(\underline{\mathrm{Set}}/y \right)/f \right\| \arrow[d, hook, "\|\mathrm{dom}\|"] \\ \mathcal{P}(y) & \simeq & \left\| \underline{\mathrm{Set}}/y \right\| \end{tikzcd}$$

$$S \subseteq y \longmapsto \left\| \begin{matrix} S \\ \downarrow {\scriptstyle i_S} \\ y \end{matrix} \right\|$$

Then in $\pi_0\left(\left(\underline{\mathrm{Set}}/Y\right)/f\right)$ all $S \subseteq f(X)$ are identified with $[f]$. The result is isomorphic to the subposet of $\mathcal{P}(Y)$ on

$$\{ S \subseteq Y \mid \exists x \in S \setminus f(X) \}.$$

Here

- $[f] \equiv \emptyset$ is the least element,
- the atoms (minimal elements above $\emptyset$) are the singletons $\{y\}$, $y \in Y \setminus f(X)$

$\boxed{\text{Claim}}$ Let $X \times_f X := \{(x_0, x_1) \mid f x_0 = f x_1\}$,

$$\Delta X := \{(x, x) \mid x \in X\} \subseteq X \times_f X .$$

$$\begin{tikzcd}
\Phi(\Delta X) \arrow[d, hook] & \left\| {}^{\mathrm{Par}\left(\left(\underline{\mathrm{Set}}/Y\right)/f \right)} \big/ (\mathrm{id}_f, \mathrm{id}_f) \right\| \arrow[d, "\|\mathrm{dom}\|"] \\
\Phi(X \times_f X) \arrow[r, phantom, "\simeq"] & \left\| \mathrm{Par}\left(\left(\underline{\mathrm{Set}}/Y\right)/f \right) \right\|
\end{tikzcd}$$

$$S \subseteq X \times_f X \longmapsto \left\| \begin{tikzcd}
S \arrow[r, bend left, "p_0"] \arrow[r, bend right, "p_1"'] \arrow[dr, "f p_0 = f p_1"'] & X \arrow[d, "f"] \\
& Y
\end{tikzcd} \right\|$$

Then in $\pi_1\left(\left(\underline{\mathrm{Set}}/Y\right)/f\right)$ any $S \subseteq \Delta X$ are identified with $[f]$.

The result is isomorphic to the subposet of $\mathcal{P}(X \times_f X)$ on

$$\left\{ S \subseteq X \times_f X \;\middle|\; \exists (x_0, x_1) \in S \text{ s.t. } x_0 \neq x_1 \right\}.$$

Here

- $[f] \equiv \emptyset$ is the least element,
- the atoms are the singletons $\{(x_0, x_1)\}$ with $x_0 \neq x_1$, $f x_0 = f x_1$.

$\boxed{\text{FUNCTORIALITY}}$

For $i \in \{0, 1\}$,

$$x \longmapsto \pi_i(\mathcal{C}/x)$$

extends to a functor

$$\pi_i(\mathcal{C}/\_) : \mathcal{C} \longrightarrow \underline{\mathbf{Pos}}.$$

i.e. for all $f : x \to y$, we have basepoint-preserving, order-preserving maps

$$f_* : \pi_i(\mathcal{C}/x) \longrightarrow \pi_i(\mathcal{C}/y)$$

Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor.

Then we have a natural transformation

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "\pi_!(\mathcal{C}/-)"'] & \arrow[d, phantom, "\Rightarrow \pi_! F"] & \mathcal{D} \arrow[dl, "\pi_!(\mathcal{D}/-)"] \\
& \underline{\mathrm{Pos}}. &
\end{tikzcd}$$

This is compatible with identities & composition.

In brief: $\pi_!$ defines a functor

$$\underline{\mathrm{Cat}} \longrightarrow \underline{\underline{\mathcal{C}at}}^{lax} / \underline{\mathrm{Pos}}.$$

FUNCTORIALITY, take 2

Consider a natural transformation

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd}$$

Then there are functors $\mathcal{C} \longrightarrow \underline{\mathcal{P}os}$.

defined on objects by

$$x \longmapsto \pi_i\left({\left(\mathcal{D}/Gx\right)}/{\alpha_x} \right)$$

$$x \longmapsto \pi_i\left({}^{\left(\mathcal{D}/Gx\right)}\big/_{\alpha_x} \right)$$

Recall that the elements of these posets are "obstructions to $\alpha_x : Fx \to Gx$ being split epi / mono".

$\hookrightarrow$ $\boxed{\text{Naturality of } \alpha \text{ induces a functorial "flow" in the invariants associated to its components}}$

EXISTENCE OF JOINS

Proposition

Let $\mathcal{C}$ be a category, κ a cardinal, $x \in \mathrm{Ob}(\mathcal{C})$.

If $\mathcal{C}$ has κ-small coproducts, then $\pi_0(\mathcal{C}/x)$ and $\pi_1(\mathcal{C}/x)$ have κ-small joins.

(Meets are w.i.p...)

MONOIDAL STRUCTURE

In general, there is no monoid structure on $\pi_0(\mathcal{C}_{/x})$.

Under certain conditions, however, a monoidal category structure $(\mathcal{C}, \otimes, i)$ together with an internal monoid structure $(m : x \otimes x \longrightarrow x,\ e : i \longrightarrow x)$ can induce it.

$\boxed{\text{FACT 1}}$

The poset reflection $\|-\|: \underline{\mathrm{Cat}} \longrightarrow \underline{\mathrm{Pos}}$ lifts to a functor $\underline{\mathrm{MonCat}} \longrightarrow \underline{\mathrm{MonPos}}$, i.e. a monoidal category reflects onto a monoidal poset.

$\boxed{\text{FACT 2}}$

A monoid object $(m : x \otimes x \longrightarrow x,\ e : i \longrightarrow x)$ determines a monoidal structure $(\otimes_m, e)$ on $\mathcal{C}/x$ and on $\mathrm{Par}(\mathcal{C}/x)$; the domain fibration is monoidal.

$\Longrightarrow$ Given a monoid (x, m, e) in $(\mathcal{C}, \otimes, i)$ we get an order-preserving homomorphism

$$\begin{array}{c} \|(\mathcal{C}/x, \otimes_m, e)\| \\ \Big\downarrow {\scriptstyle \|\mathrm{dom}\|} \\ \|(\mathcal{C}, \otimes, i)\| \end{array}$$

However, in general, the pushout defining $\pi_0(\mathcal{C}/_x)$ in <u>Pos</u> does NOT lift to a pushout (cokernel) in <u>MonPos</u> ...

↳ This is only true if the equivalence relation generated by the image of $\|dom\|$ is a congruence for multiplication in $\|\mathcal{C}\|$

Lemma TFAE:

- $\pi_0(\mathcal{C}/x)$ admits a monoidal structure, s.t. $\|\mathcal{C}\| \twoheadrightarrow \pi_0(\mathcal{C}/x)$ is the cokernel of $\|\mathrm{dom}\|$;
- for all y, z s.t. $\|y\| \leq \|x\|$, either $\|z\| \leq \|x\|$ or
$$\|z\| = \|y\| \cdot \|z\| = \|z\| \cdot \|y\| .$$

(In particular: $\forall y$ either $\|y\| \leq \|x\|$ or
$$\|y\| = \|x\| \cdot \|y\| = \|y\| \cdot \|x\|)$$

There is a similar story for the π_1, starting from an order-preserving homomorphism

$$\begin{array}{c} \left\| \mathrm{Par}(\mathcal{C}/x) \Big/ \left(x \overset{\mathrm{id}_x}{\underset{\mathrm{id}_x}{\rightrightarrows}} x \right) \right\| \\ \Big\downarrow \|\mathrm{dom}\| \\ \left\| \mathrm{Par}(\mathcal{C}/x) \right\| \end{array}$$

...

But it turns out there's a simpler "sufficient, almost necessary" condition ...

Def $x \in Ob(\mathcal{C})$ is a *weak zero object* if $\forall y \in Ob(\mathcal{C})$ $\exists$ $x \rightleftarrows y$

Proposition (x, m, e) monoid in $(\mathcal{C}, \otimes, i)$

1. If $e : i \to x$ is a *weak zero object in $\mathcal{C}/_x$*, then $\pi_1(\mathcal{C}/_x)$ has a monoidal structure, s.t. $\|Par(\mathcal{C}/_x)\| \twoheadrightarrow \pi_1(\mathcal{C}/_x)$ is the cokernel of $\|dom\|$.
2. If $\exists\, s : i \to x$ s.t. $s \neq e$, then the converse implication also holds.

Theorem (x, m, e) monoid in $(\mathcal{C}, \otimes, i)$

① If $e : i \to x$ is a weak zero object in $\mathcal{C}/x$,

then a) $\|\mathcal{C}\| \twoheadrightarrow \pi_0(\mathcal{C}/x)$ is an isomorphism,

b) $\|\mathrm{Par}(\mathcal{C}/x)\| \twoheadrightarrow \pi_1(\mathcal{C}/x)$ is a cokernel ;

② If $\exists\, s : i \to x$, $s \neq e$, the converse implication also holds.

$\boxed{\text{Example}}$

If (x, m, e) is a monoid in a monoidal groupoid $(\mathcal{G}, \otimes, i)$, then e is always a weak zero object in $\mathcal{G}/x$.

$\boxed{\text{So we get monoid structures on } \pi_0(\mathcal{G}, x),\ \pi_1(\mathcal{G}, x)}$

$\boxed{\text{Example}}$

If $(\mathcal{C}, \oplus, 0)$ is semicocartesian, i.e.

0 is initial, then

① 0 has a "trivial" monoid structure

$$0 \oplus 0 \xrightarrow{\sim} 0, \qquad 0 \xrightarrow{\mathrm{id}_0} 0$$

② id_0 is a zero object in $\mathcal{C}/0$

So the conditions are satisfied for

$$\pi_0(\mathcal{C}/0), \qquad \pi_1(\mathcal{C}/0)$$

$\boxed{\text{Example: } (\underline{\mathrm{Ring}}, \otimes, \mathbb{Z})}$

In general, given a monoidal category $(\mathcal{C}, \otimes, i)$, the slice ${}^{i}/\mathcal{C}$ is always cocartesian monoidal with unit $i \xrightarrow{\mathrm{id}_i} i$

$$\Rightarrow \boxed{\begin{array}{c} \pi_0\left(({}^{i}/\mathcal{C})/_{\mathrm{id}_i}\right), \quad \pi_1\left(({}^{i}/\mathcal{C})/_{\mathrm{id}_i}\right) \\ \text{have monoidal structures} \end{array}}$$

THANK YOU FOR YOUR ATTENTION!

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