Diagrammatic sets
between rewriting and topology

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Reference

arXiv:1909.07639

Representable diagrammatic sets as a model of weak higher categories.

Dimensions in rewriting

- Dimension 0: (labelled) abstract rewrite system

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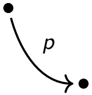

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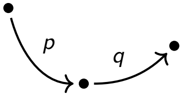

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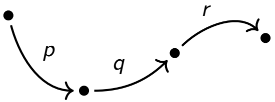

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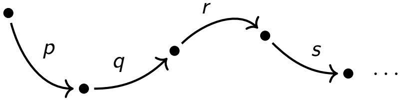

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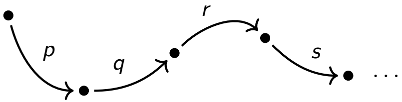

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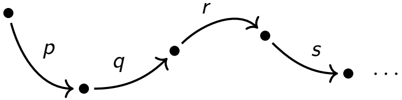

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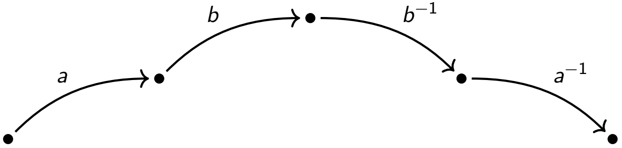

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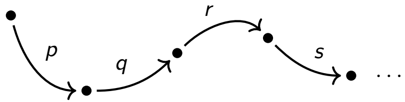

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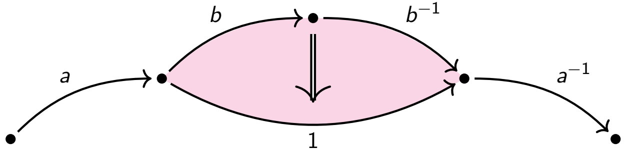

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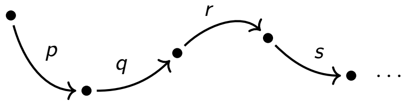

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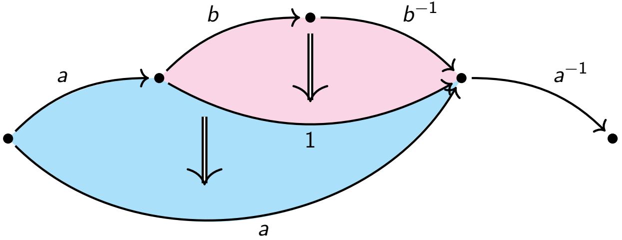

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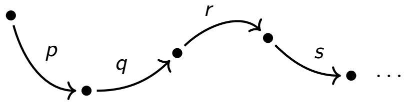

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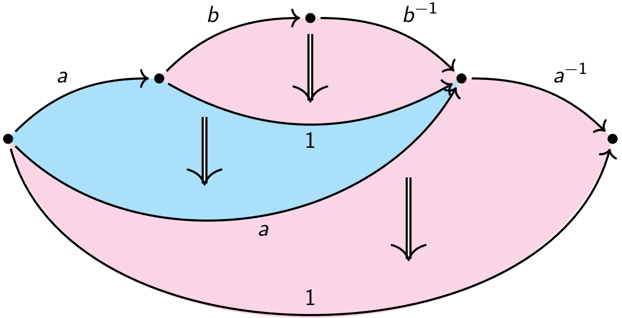
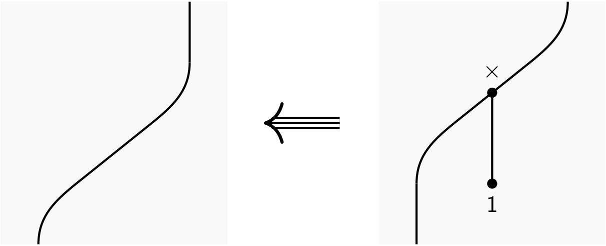


Dimensions in rewriting

- Dimension 2: algebraic theories / monoidal categories



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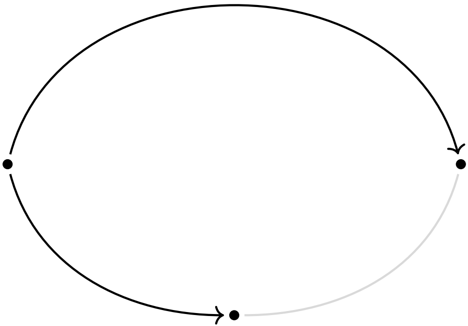

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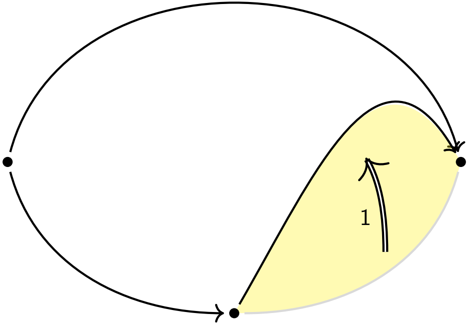

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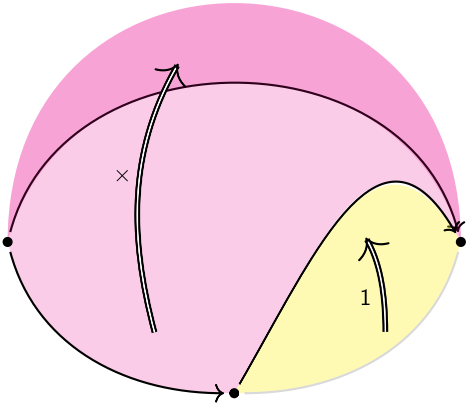

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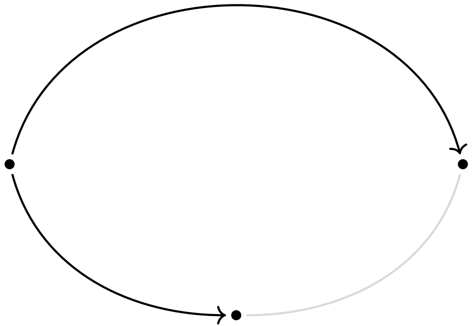

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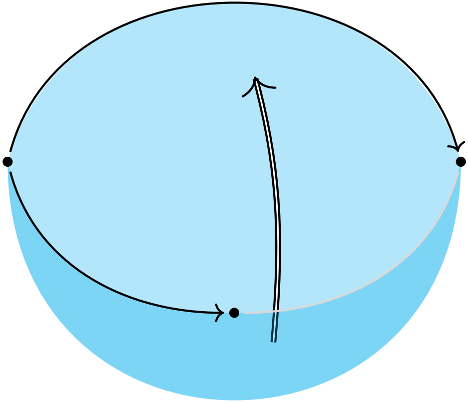

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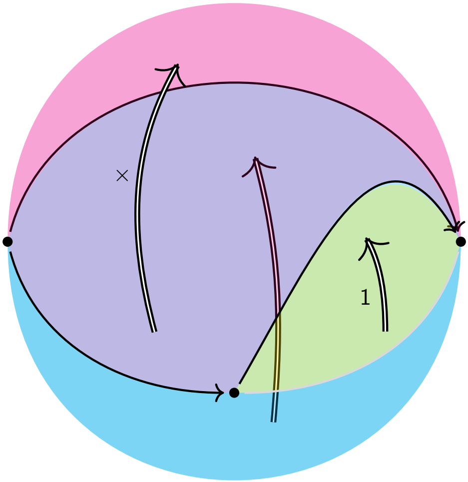

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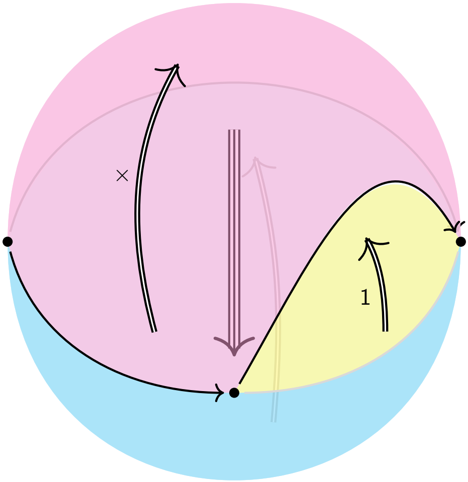

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Rewriting objects of dimension > 2:

- higher/homotopical algebra
- k-manifolds in n-space (e.g. k=1, n=3: knots and braids)

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Higher-dimensional "rewrites" for a fixed base dimension:

- confluence, coherence...

The key observation

A higher-dimensional rewrite system is like a **CW complex**, a space built by pasting together topological balls (**cells**)

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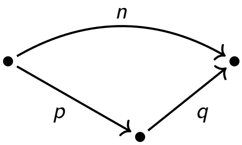

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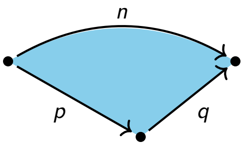

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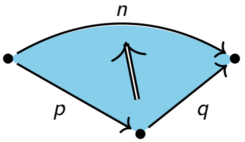

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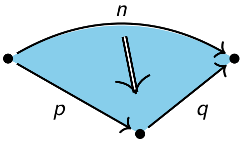

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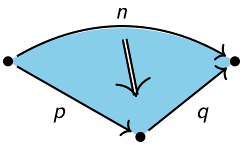

"Point-set" cells not available — need a combinatorial notion of

1. **directed cells**
2. their **pasting**

Polygraphs

Standard framework for higher-dimensional rewriting:

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The pasting maps are **not sound** for the interpretation of "rewrite systems as CW complexes"

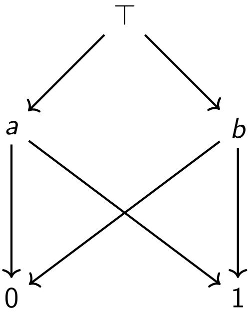
...plus other technical issues

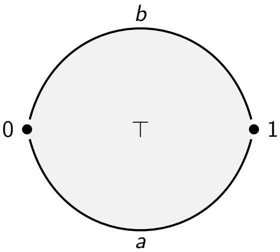
Towards diagrammatic sets

Let the CW complex interpretation
guide the choice of a framework

Face posets

We can associate to a CW complex its face poset...





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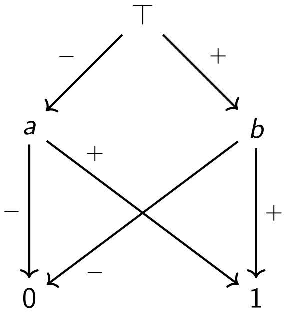

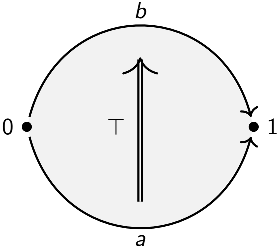

and to a pasting diagram its oriented face poset.

Face posets

Regular CW complex X: pasting maps are homeomorphisms with their image

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A classical theorem of combinatorial topology

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Conjecture

A regular pasting diagram is specified up to cellular isomorphism by its oriented face poset

Diagrammatic sets

- Directed n-cells are modelled by regular directed complexes
  (which are oriented face posets of regular pasting diagrams)

  with a greatest element of rank n
  (so the underlying poset is the face poset of a regular CW n-ball)

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- Pasting is given by maps of posets that are compatible functorially with both realisations

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  $\rightarrow$ “nullary” operations in universal algebra

Enough for higher-dimensional rewriting?

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Other features:

- Some constructions which are a nightmare with strict ω-categories (**lax Gray products**, **joins**) are easy with diagrammatic sets

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- Good geometric realisation; can be used to construct CW complexes

  (in fact, diagrammatic sets satisfy a version of the *homotopy hypothesis* — one can reason about spaces/homotopy types in terms of their diagrammatic nerve, as with simplicial sets)

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We want to interpret higher-dimensional rewrite systems in a model of higher categories

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We want to interpret higher-dimensional rewrite systems in a model of higher categories

Diagrammatic sets have *some* features of a model (pasting, units)

Idea: higher categories $\to$ diagrammatic sets with an **internal** notion of *weak composition*

(in the spirit of categorical semantics:
syntax and semantics in the same universe)

Equivalences and weak composition

Computational meaning of composition:

A diagram x can be **substituted** in every context with a cell $[\![x]\!]$

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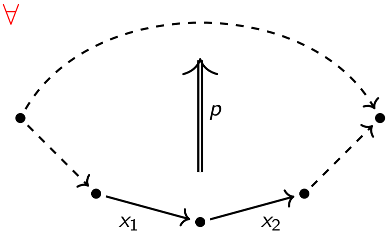
$\downarrow$

There are special equivalence cells $x \Rightarrow y$, $y \Rightarrow x$, which mediate between all cells containing x and all cells containing y in their boundary

Equivalences and weak composition



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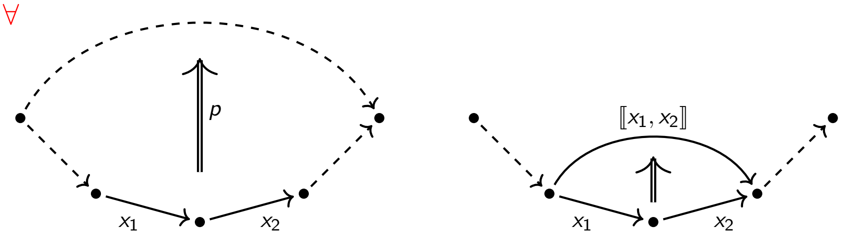

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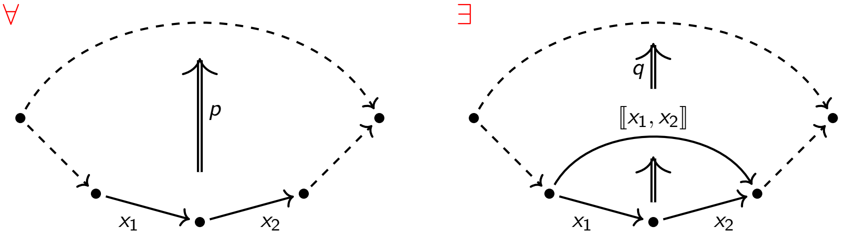

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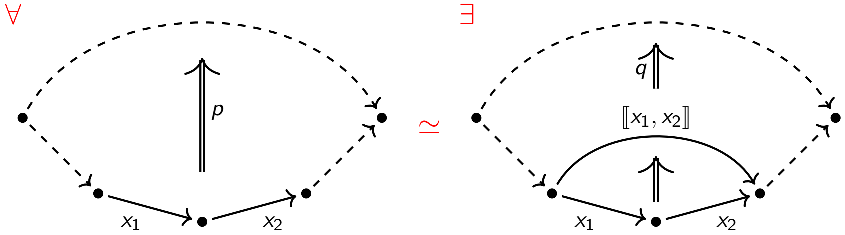

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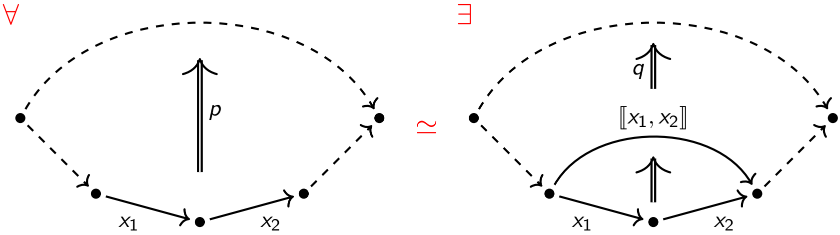

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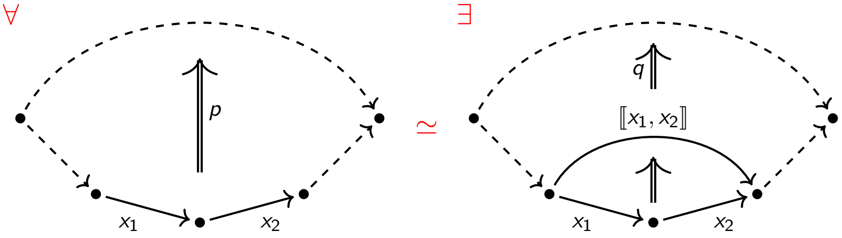

And this equivalence should be witnessed by **3-dimensional equivalence cells**...

whose definition involves 4-dimensional equivalence cells, etc

Equivalences and weak composition

This is a **coinductive** definition.

Let X be a diagrammatic set. For all subsets $A \subseteq \mathrm{Cell}(X)$, define

$$\begin{aligned}\mathcal{F}(A) := \{x : U \to X \mid \; & \text{for all } \alpha \in \{+,-\} \text{ and} \\ & (\Lambda \hookrightarrow W, \lambda : \Lambda \to X) \in \mathcal{D}iv(x, \partial^\alpha U), \\ & \text{there exists } (h : W \to X) \in A \text{ such that } h|_\Lambda = \lambda\};\end{aligned}$$

Then $\mathcal{F}$ is an order-preserving map on $\mathcal{P}(\mathrm{Cell}(X))$. Its **greatest fixed point** is the set $\mathcal{E}qX$ of equivalence cells of X.

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Proof method: if $A \subseteq \mathcal{F}(A)$, then $A \subseteq \mathcal{E}qX$.

Equivalences and weak composition

Representable diagrammatic set (RDS)

A diagrammatic set where, for all diagrams x, there exist cells $[\![x]\!], [\![x]\!]'$ and equivalence cells $x \Rightarrow [\![x]\!], [\![x]\!]' \Rightarrow x$.

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Theorem

In a representable diagrammatic set,

1. all degenerate cells are equivalence cells, and
2. $\simeq$ is an equivalence relation.

Representable diagrammatic sets

1. "**Groupoidal**" RDSs (in which every cell is an equivalence) model all homotopy types.
2. Conditional to the conjecture on regular pasting diagrams, strict ω-categories **embed** as a full subcategory (if one takes morphisms that preserve a choice of weak composites)
3. There are n-**truncated** RDSs corresponding to weak n-categories. 2-truncated RDSs are equivalent to bicategories