Locally Cubical Gray Categories

Edward Morehouse

Tallinn University of Technology

2022-12-19

Overview

Gray defined a monoidal product for 2-categories that is left adjoint to the internal hom,

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We use this to give an algebraic presentation of the 3-dimensional structure of double categories and their morphisms, and consider the one-object case, which endows double categories with Gray-monoidal structure.

Double Categories

A *double category* [Ehr62] is a category internal to $\textsc{Cat}$,

$$
\begin{tikzcd}[column sep=huge]
\mathbb{C}_1 \times_{\mathbb{C}_0} \mathbb{C}_1 \arrow[r, bend left=50, "\pi_0"] \arrow[r, "- \odot -"] \arrow[r, bend right=50, "\pi_1"'] & \mathbb{C}_1 \arrow[r, bend left=50, "L"] \arrow[r, bend right=50, "R"'] & \mathbb{C}_0 \arrow[l, "U"']
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0-cells or “objects”, $\quad A,$

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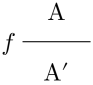
0-cells or “objects”, $\quad A, A'$,
vertical 1-cells or “arrows”, $\quad f : A \to A'$,



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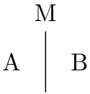

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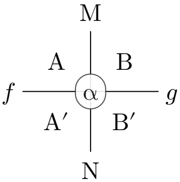
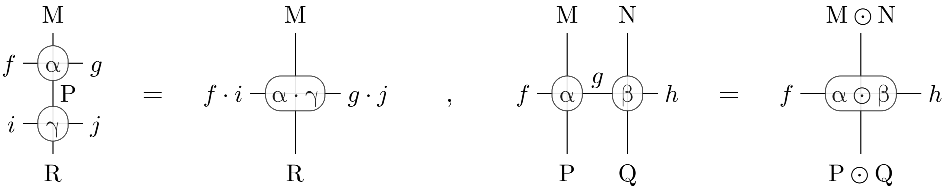
horizontal 1-cells or “proarrows”, $\quad M : A \nrightarrow B\,, N : A' \nrightarrow B',$
2-cells or “squares”, $\quad \alpha : {}^{M}_{f}\Diamond^{g}_{N}.$



Composition in Double Categories

Squares compose by pasting in both dimensions [Mye16]:



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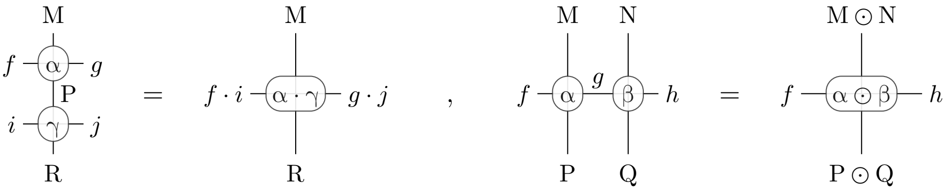
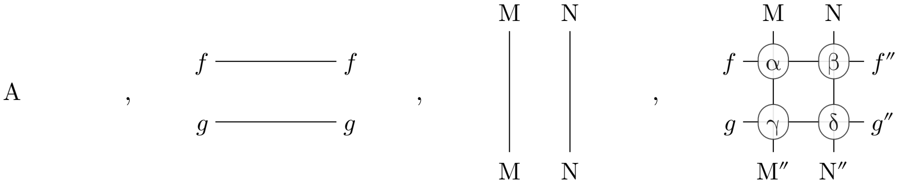


By functoriality of U and $\odot$, these squares are well-defined:



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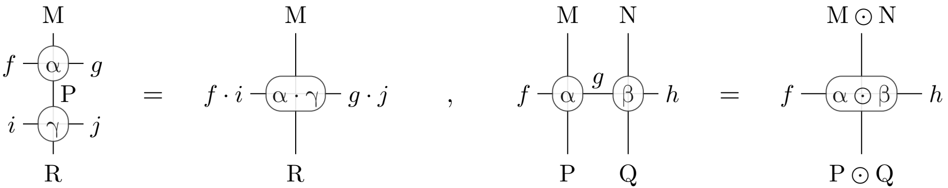

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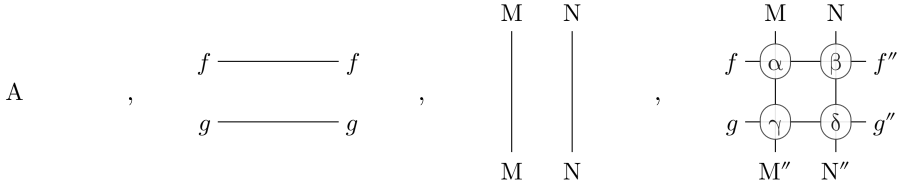
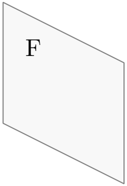


Technically, our double categories are *weak*, but coherence lets us pretend they are *strict*. [GP99]

Morphisms of Double Categories

There is a hierarchy of morphisms of double categories:

functors, $\mathrm{F} : \mathbb{C} \to \mathbb{D}$,

$\mathbb{C}$  $\mathbb{D}$

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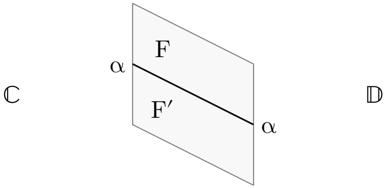

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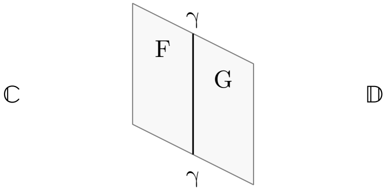

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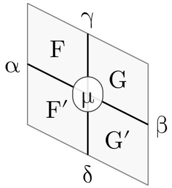
$\mathbb{C}$  $\mathbb{D}$

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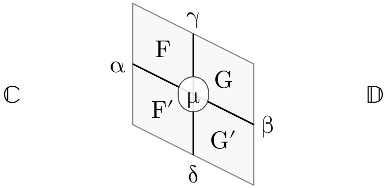

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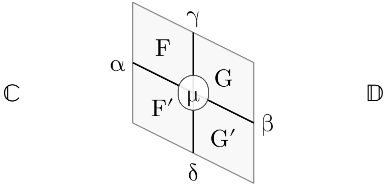

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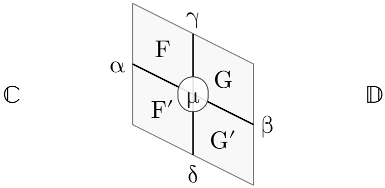

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Using Böhm's *Gray tensor product of double categories* [Böh19] we do the same thing in the cubical setting.

Locally Cubical Gray Categories – n-cells

A *locally cubical Gray category* $\mathbb{C}$ has

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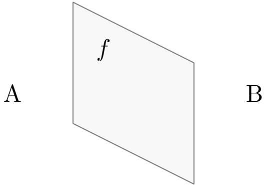

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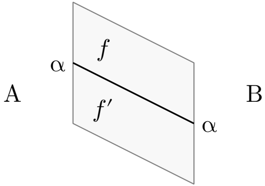

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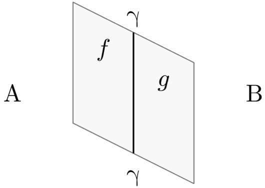

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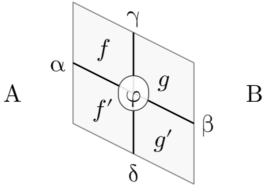
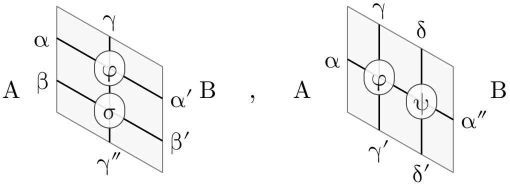
vertical 2-cells, $\alpha : f \to f'$, $\beta : g \to g'$,
horizontal 2-cells, $\gamma : f \nrightarrow g$, $\delta : f' \nrightarrow g'$,
3-cells, $\varphi : {}^{\gamma}_{\alpha}\Diamond^{\beta}_{\delta}$.



Locally Cubical Gray Categories – local composition

For each pair of 0-cells we have a *hom double category* $\mathbb{C}\,(\mathrm{A} \to \mathrm{B})$.



Locally Cubical Gray Categories – principal composition

For $m, n \in \mathbb{N}$ with $m + n \leq 2$,
composing an $(m+1)$-cell with 0-cell boundary $\mathrm{A} \to \mathrm{B}$
with an $(n+1)$-cell with 0-cell boundary $\mathrm{B} \to \mathrm{C}$
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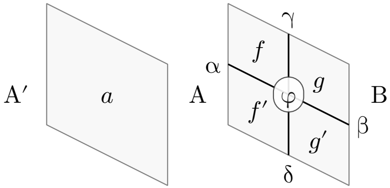
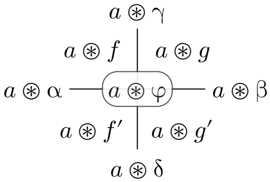
We read off the boundaries of composite cells
from the projection string diagram of a surface diagram.

Whiskerings

When $m = 0$ or $n = 0$ the principal composition is called *whiskering* $(- \circledast -)$.





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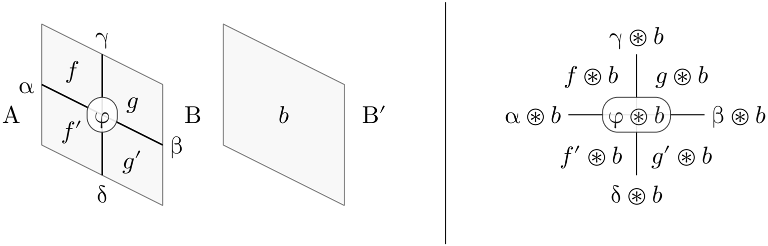

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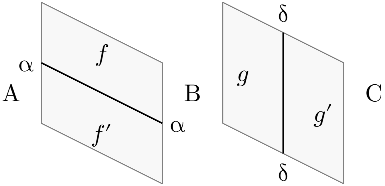
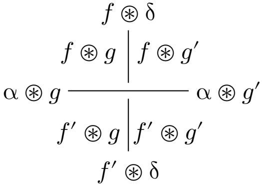
When $m = n = 1$ the principal composition is called *interchange* $(\chi_{(-,-)})$.





$= \quad \chi_{(\alpha,\delta)}$

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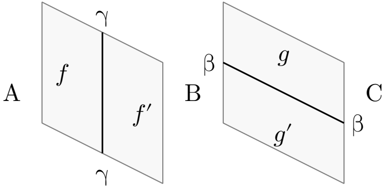

$$\begin{array}{ccc}
 & \gamma \circledast g & \\
f \circledast g & \Big| & f' \circledast g \\
f \circledast \beta \;\text{———} & \Big| & \text{———}\; f' \circledast \beta \\
f \circledast g' & \Big| & f' \circledast g' \\
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\quad = \quad \chi_{(\gamma,\beta)}$$

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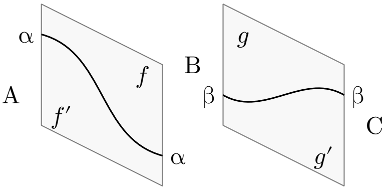

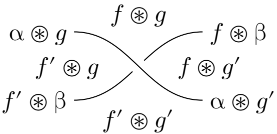

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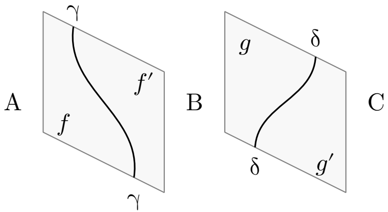

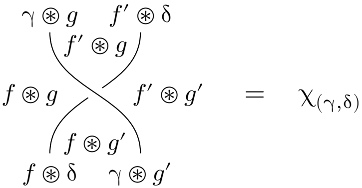

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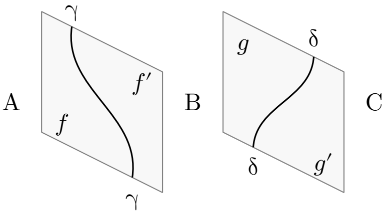
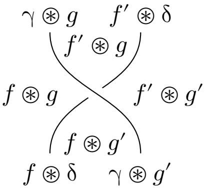




$$= \chi_{(\gamma,\delta)}$$

This variance for homogeneous interchangers is called “oplax”, and its opposite “lax”.

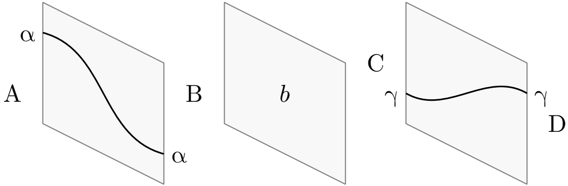
Principal Composition Laws

Principal composition is strictly unital and associative,
and compatible with local composition in hom double categories.

We can “read off” laws from diagrams without critical points.

E.g.

$$\chi_{(\alpha \circledast b, \gamma)} = \chi_{(\alpha, b \circledast \gamma)}$$



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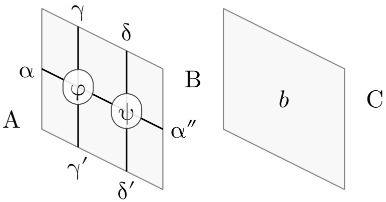

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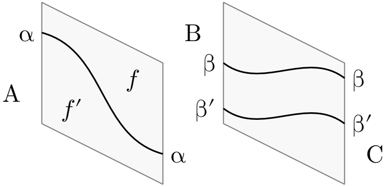
$$\chi_{(\alpha,\beta\cdot\beta')} = (\chi_{(\alpha,\beta)} \cdot \mathrm{U}(f' \circledast \beta')) \odot (\mathrm{U}(f \circledast \beta) \cdot \chi_{(\alpha,\beta')})$$



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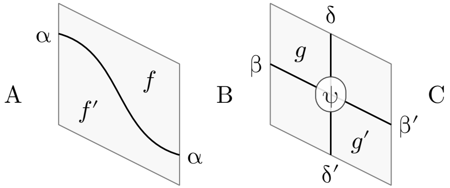
We don't get a *structure* by composing a 2-cell with a 3-cell because there are no 4-cells.



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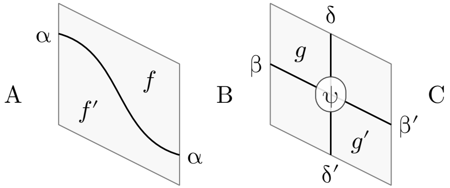
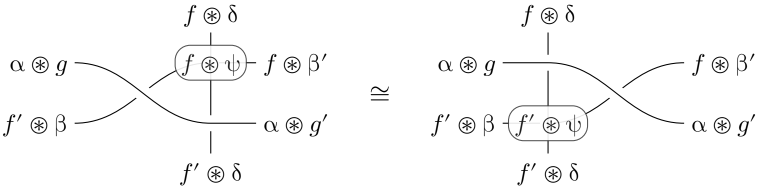


Instead we get the *property* of a naturality equation.
We can read these off of diagrams by perturbing them away from critical points [Mor22].



Locally Cubical Gray Categories of Interest

Proposition

There is a locally cubical Gray category where

0-cells are double categories,
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Proposition

A (locally globular) Gray category is a locally cubical Gray category with trivial horizontal 2-cells.

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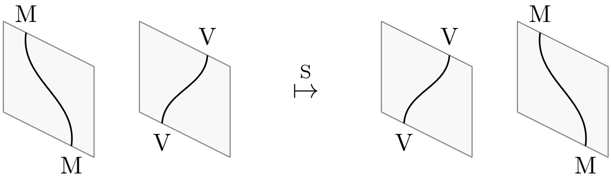
Double category $\mathbb{C}$ is Gray-monoidal if functors $\otimes_{\mathbb{C}} : \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$ and $\mathrm{I}_{\mathbb{C}} : \mathbb{1} \to \mathbb{C}$
form a monoid.

Braiding

The *swap functor* $\mathrm{S}_{(\mathbb{C},\mathbb{D})} : \mathbb{C} \otimes \mathbb{D} \to \mathbb{D} \otimes \mathbb{C}$ reverses ordered pairs.



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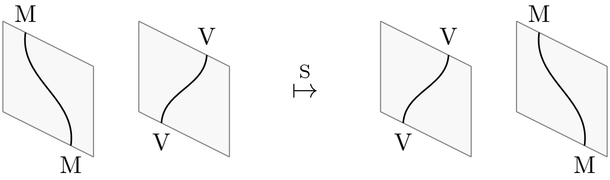
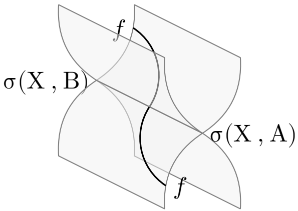


A *braiding* for a Gray-monoidal double category with invertible interchangers $\mathbb{C}$ is a vertical pseudo transformation $\sigma : (\mathbb{C} \otimes \mathbb{C} \to \mathbb{C})\,(\otimes_{\mathbb{C}} \to S_{(\mathbb{C},\mathbb{C})} \cdot \otimes_{\mathbb{C}})$



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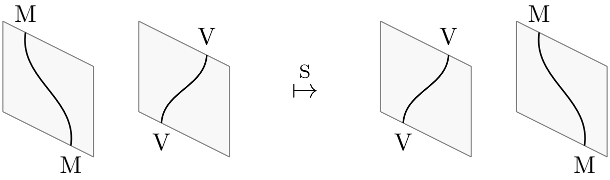

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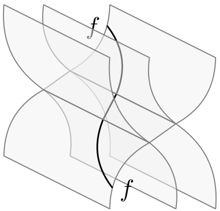

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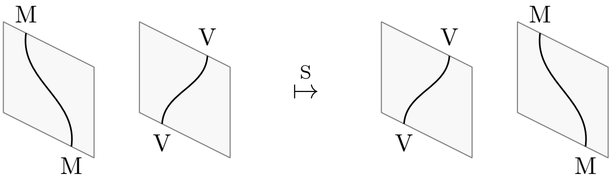

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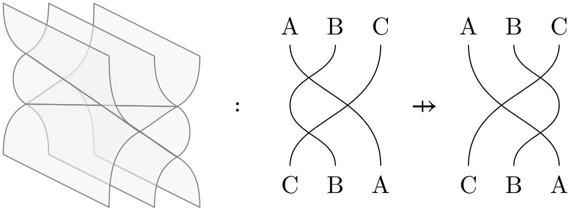
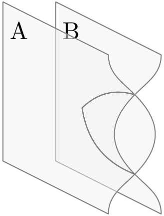


Syllepsis

A *syllepsis* for a braided Gray-monoidal double category $\mathbb{C}$
is an invertible globular modification $\upsilon : (\otimes_{\mathbb{C}} \to \otimes_{\mathbb{C}})(\mathrm{id}(\otimes_{\mathbb{C}}) \nrightarrow \sigma \cdot (S \cdot\cdot\, \sigma))$
relating the unbraiding to a pair of consecutive braidings



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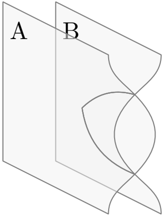

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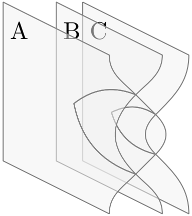

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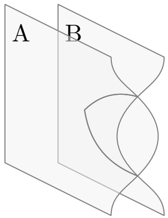

that is coherent for monoidal composition [DS97]. A syllepsis is a *symmetry*
if it is the unit of an adjoint equivalence $\sigma \dashv \mathrm{S} \cdot\cdot\, \sigma$.

Wrap-Up

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Preprint available: *Cartesian Gray-Monoidal Double Categories*

`https://www.ioc.ee/~ed/` (arXiv version coming soon)

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