Homework 2:
Unitary Evolution and Tensor Products

Quantum Information Systems   Wesleyan University

due 2017.03.29

Problem 1
Consider the linear operator, $\mathrm{K}_{|0\rangle} : \mathbb{C}^2 \to \mathbb{C}^2$,

$$|0\rangle \longmapsto |0\rangle \qquad \text{and} \qquad |1\rangle \longmapsto |0\rangle$$

a. Write out the matrix representation of this operator.

b. Write out the outer product representation of this operator.

c. Check that your two answers above are consistent with one another by explicitly multiplying out the outer product and showing that it agrees with the matrix.

d. Prove that this operator is *not* unitary.

Problem 2
In class we met the phase operator for the Z-basis:

$$\mathrm{Z}(\varphi) \quad = \quad |0\rangle\langle 0| \; + \; e^{i\varphi}|1\rangle\langle 1|$$

There is also a phase operator for the X-basis:

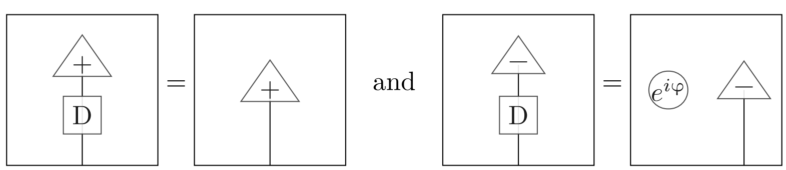
$$\mathrm{X}(\varphi) \quad = \quad |+\rangle\langle +| \; + \; e^{i\varphi}|-\rangle\langle -|$$

a. Write an expression for this operator that uses only the single-qubit operators that we introduced in class (i.e. identity, negation, Hadamard and Z-phase).

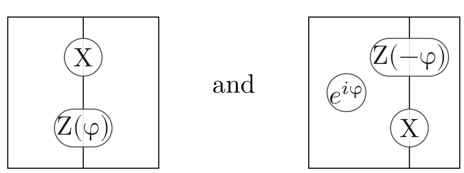
b. Explain how you know that your proposed expression represents an operator that is *unitary.*

c. Draw the diagram corresponding to your proposed expression.

d. Assuming your diagram is called "D", use *diagram rewriting* to prove that



Problem 3
In this problem we are going to discover an interaction law of two single-qubit operators. Consider the two composite single-qubit operators:



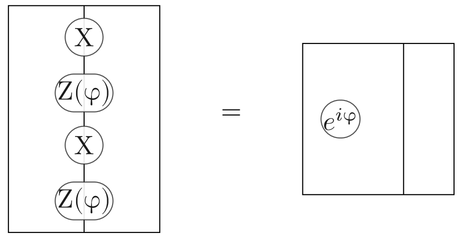
where "X" is the Z-negation operator ($|1\rangle\langle 0| + |0\rangle\langle 1|$) and "Z(φ)" is the Z-phase phase operator described in problem 2.

a. Explicitly compute the matrix representing each of the composite operators.

b. What do you observe about these two matrices?

c. Write down an equation involving the composite operators recording your observation.

d. Use your equation and diagram rewriting to show that $\mathrm{X} \cdot \mathrm{Z}(\varphi)$ is an *involution*, up to a phase:



Problem 4
Given the following column vectors:

$$\mathrm{A} := \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \qquad \mathrm{B} := \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \qquad \mathrm{C} := \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

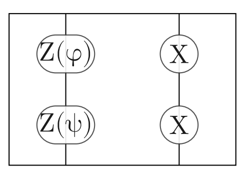
explicitly compute each of the *Kronecker products*:

- $(\mathrm{A} \otimes \mathrm{B}) \otimes \mathrm{C}$
- $\mathrm{A} \otimes (\mathrm{B} \otimes \mathrm{C})$

and observe that they are equal, consistent with our claim of *tensor associativity*.

Problem 5
Consider the following diagram describing an evolution of a 2-qubit system:

E :=



a. Write out an algebraic expression describing this evolution. I.e., use the language consisting of tensor product $(- \otimes -)$, and either forward $(- \cdot -)$ or reverse $(- \circ -)$ composition (your choice), together with the single-qubit unitary operators to write an expression "E = ..." describing the sequence of evolutionary steps depicted in the diagram.

b. Write out the matrix expression corresponding to your algebraic expression from part (a).

c. Use matrix arithmetic (matrix multiplication and Kronecker product) to evalutate your matrix expression from part (b).

d. Use diagram rewriting to simplify the diagram for E to an equivalent evolution containing a minimal number of single-qubit operations.

e. Compute the matrix representation of your simplified diagram from part (d) and confirm that it agrees with the matrix you computed for part (c).

Which method of reasoning do you prefer, matrix arithmetic or diagram rewriting?