Homework 3:
Multi-Qubit Operators and States

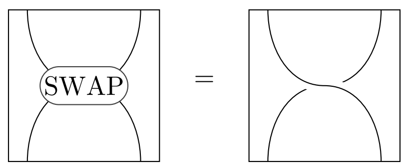
Quantum Information Systems   Wesleyan University

due 2017.04.05

Problem 1 (the SWAP operator)
In class we met the two-qubit operator SWAP:

$$\mathrm{SWAP} \quad = \quad |00\rangle\langle 00| \; + \; |10\rangle\langle 01| \; + \; |01\rangle\langle 10| \; + \; |11\rangle\langle 11|$$

which exchanges the states of two qubits. As a mnemonic for this fact, we gave SWAP the graphical representation of "crossing wires":

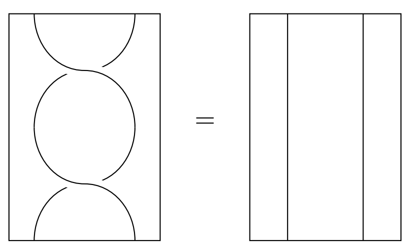


Here, we explore some geometric properties of this graphical representation.

a. Use matrix arithmetic to verify that SWAP is an *involution*:

$$\mathrm{SWAP} \cdot \mathrm{SWAP} \quad = \quad \mathrm{Id}_{\mathbb{C}^2 \otimes \mathbb{C}^2}$$

or diagrammatically:



Since SWAP is its own inverse, the wires in a diagram cannot become tangled. There is no difference between "crossing over" and "crossing under" so we will draw them interchangeably.

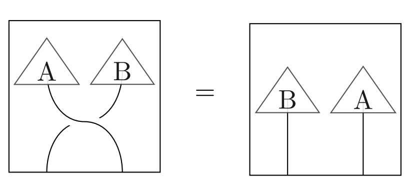
b. Given two single-qubit states with matrix representations,

$$|\mathrm{A}\rangle \quad := \quad \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \qquad \text{and} \qquad |\mathrm{B}\rangle \quad := \quad \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

use matrix arithmetic to verify that we can "comb out a swap":

$$(|\mathrm{A}\rangle \otimes |\mathrm{B}\rangle) \cdot \mathrm{SWAP} \quad = \quad |\mathrm{B}\rangle \otimes |\mathrm{A}\rangle$$

or diagrammatically:



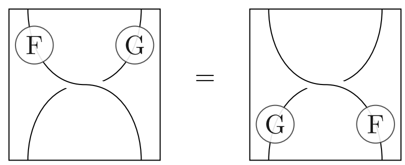
c. Given two single-qubit operators with matrix representations,

$$\mathrm{F} \quad := \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \qquad \mathrm{G} \quad := \quad \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

use matrix arithmetic to verify that they can be "pushed across a swap":

$$(\mathrm{F} \otimes \mathrm{G}) \cdot \mathrm{SWAP} \quad = \quad \mathrm{SWAP} \cdot (\mathrm{G} \otimes \mathrm{F})$$

or diagrammatically:

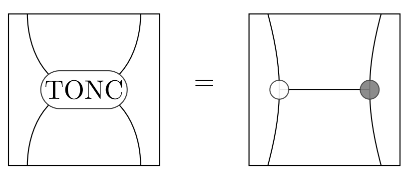


Problem 2 (the CNOT operator)
In class we also met the two-qubit operator CNOT, in which the first qubit "controls" the negation of the second qubit. Now we consider the symmetric situation, in which the second qubit "controls" the negation of the first qubit instead.

Here is a "truth table" for the the two-qubit operator TONC (i.e. CNOT spelled backward):

A	B	TONC(A , B)	
0	0	0	0
0	1	1	1
1	0	1	0
1	1	0	1

In the graphical notation, we will represent TONC as the mirror-image of CNOT:



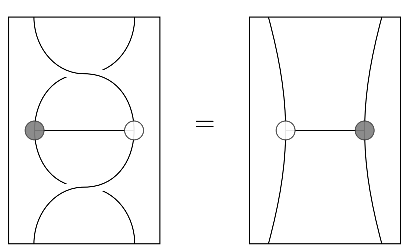
a. Write an outer-product representation of the TONC operator.

b. Write the matrix representation of the TONC operator.

c. Use matrix arithmetic to verify the (perhaps obvious) identity:

$$\mathrm{SWAP} \cdot \mathrm{CNOT} \cdot \mathrm{SWAP} \quad = \quad \mathrm{TONC}$$

or diagrammatically:



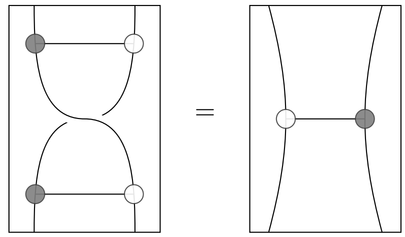
d. Use this identity and diagram rewriting to show that:

$$\mathrm{TONC} \cdot \mathrm{SWAP} \quad = \quad \mathrm{SWAP} \cdot \mathrm{CNOT}$$

e. Use matrix arithmetic to verify the (perhaps less-obvious) identity:

$$\text{CNOT} \cdot \text{SWAP} \cdot \text{CNOT} \quad = \quad \text{TONC}$$

or diagrammatically:



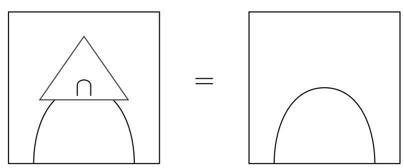
f. Use this identity and diagram rewriting to show that:

$$\text{CNOT} \cdot \text{TONC} \quad = \quad \text{SWAP} \cdot \text{CNOT}$$

Problem 3 (the entangled state $|\cap\rangle$)
We also met the two-qubit entangled state $|\cap\rangle$:

$$|\cap\rangle \quad = \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

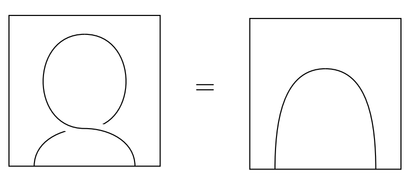
for which measurement of the two qubits always results in the same (totally unpredictable) outcome. As a mnemonic for this fact, we gave $|\cap\rangle$ the graphical representation of a "bent wire":



a. Use matrix arithmetic to verify that $|\cap\rangle$ is *co-commutative*:

$$|\cap\rangle \cdot \text{SWAP} \quad = \quad |\cap\rangle$$

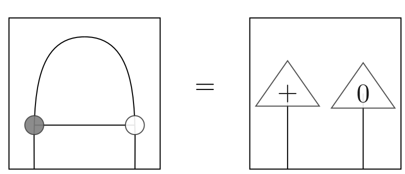
or diagrammatically:



b. Use matrix arithmetic to verify the identity:

$$|\cap\rangle \cdot \text{CNOT} \quad = \quad |+\rangle \otimes |0\rangle$$

or diagrammatically:



c. Use this identity and diagram rewriting to show that:

$$|\cap\rangle \cdot \text{TONC} \quad = \quad |0\rangle \otimes |+\rangle$$