Homework 6:
The ZX-Calculus

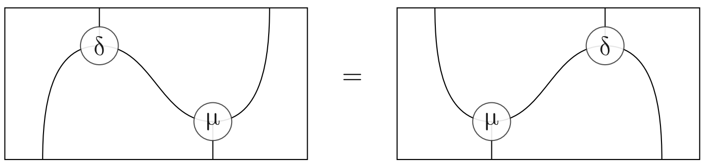
Quantum Information Systems   Wesleyan University

due 2017.05.08

Problem 1 (the Frobenius law)
Recall from lecture that the **Frobenius law** states:

$$(\delta \otimes \mathrm{Id}) \cdot (\mathrm{Id} \otimes \mu) \quad = \quad (\mathrm{Id} \otimes \delta) \cdot (\mu \otimes \mathrm{Id})$$

or diagramatically,



Recall also that the Hilbert space interpretations of Z- and X- basis spiders are given by:

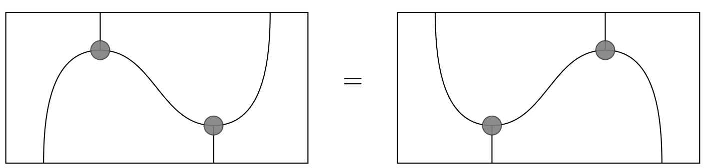
$$[\![\mathrm{Z}^n_m(\varphi)]\!] \quad := \quad |0^m\rangle\langle 0^n| \quad + \quad e^{i\varphi}(|1^m\rangle\langle 1^n|)$$
$$[\![\mathrm{X}^n_m(\varphi)]\!] \quad := \quad |+^m\rangle\langle +^n| \quad + \quad e^{i\varphi}(|-^m\rangle\langle -^n|)$$

a. Write the matrix representation for the spiders Z^1_2 and Z^2_1.

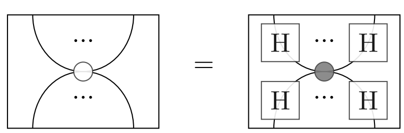
b. Use matrix arithmetic to show that Z-basis spiders with phase 0 satisfy the *Frobenius law*:

$$[\![(\mathrm{Z}^1_2 \otimes \mathrm{Id}) \cdot (\mathrm{Id} \otimes \mathrm{Z}^2_1)]\!] \quad = \quad [\![(\mathrm{Id} \otimes \mathrm{Z}^1_2) \cdot (\mathrm{Z}^2_1 \otimes \mathrm{Id})]\!]$$

or diagrammatically:



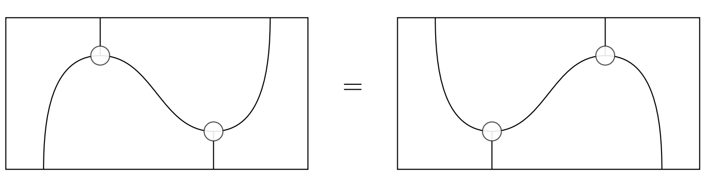
c. Use the equation in part (b) together with the **color-change law**:



to show by diagram rewriting that X-basis spiders with phase 0 satisfy the Frobenius law:

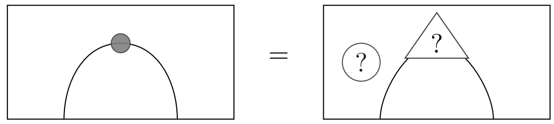
$$[\![(\mathrm{X}^1_2 \otimes \mathrm{Id}) \cdot (\mathrm{Id} \otimes \mathrm{X}^2_1)]\!] \quad = \quad [\![(\mathrm{Id} \otimes \mathrm{X}^1_2) \cdot (\mathrm{X}^2_1 \otimes \mathrm{Id})]\!]$$

or diagrammatically:



Problem 2 (Frobenius implies zigzag)

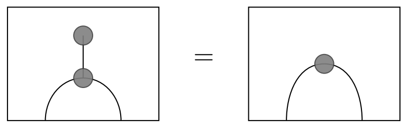
a. Write an outer product representation for the interpretation of the spider Z_2^0.
Up to a scalar, which familiar state is this? Record this result in the form of a diagrammatic equation by filling in the missing right-hand side below:



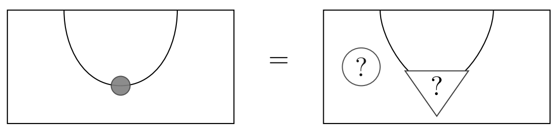
b. Using either matrix arithmetic or algebra (your choice) verify that

$$[\![Z_1^0 \cdot Z_2^1]\!] \;=\; [\![Z_2^0]\!]$$

i.e.:



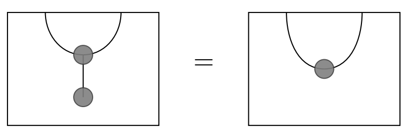
c. Write an outer product representation for the interpretation of the spider Z_0^2.
Up to a scalar, which familiar effect is this? Record this result in the form of a diagrammatic equation by filling in the missing right-hand side below:



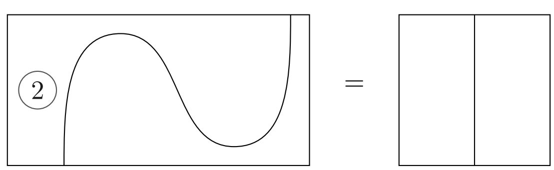
d. Using either matrix arithmetic or algebra (your choice) verify that

$$[\![Z_1^2 \cdot Z_0^1]\!] \;=\; [\![Z_0^2]\!]$$

i.e.:

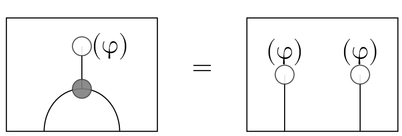


e. Use the equations above, together with those of a *Frobenius algebra* to show by diagram rewriting that the following *zigzag law* holds in the ZX-calculus:



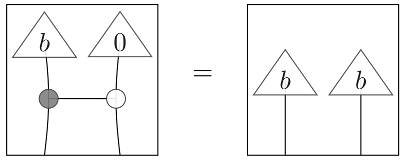
i.e. $2\,(\cap \otimes \mathrm{Id}) \cdot (\mathrm{Id} \otimes \cup) = \mathrm{Id}$. *Note*: In the reading, the bent-wire state ($\cap$) and effect ($\cup$) are not normalized, so the scalars work out slightly differently there.

Problem 3 (Z-basis copying)
Recall that the spider Z_2^1 can copy Z-basis states:



for $\varphi \in \{0\,,\pi\}$ *only*

While studying the *no-cloning theorem* we observed that the CNOT operator can also be used to copy Z-basis states:



for $b \in \{0\,, 1\}$ *only*

a. *Either* give a precise argument appealing to the properties of linear maps *or* use matrix arithmetic (your choice) to show that the interpretation of the operator $(\mathrm{Id} \otimes |0\rangle) \cdot \mathrm{CNOT}$ is the same as that of the spider Z_2^1.

b. Translate the circuit diagram for $(\mathrm{Id} \otimes |0\rangle) \cdot \mathrm{CNOT}$ into spider notation and use the rules of the ZX-calculus to rewrite it to the spider Z_2^1

Problem 4 (The GHZ state)
After the Bell states, one of the most important entangled states is the **GHZ state**:

$$|\mathrm{GHZ}\rangle \quad := \quad \frac{1}{\sqrt{2}}\,(|000\rangle + |111\rangle)$$

a. What is the spider notation for this state (you may ignore the scalar)?

b. Use the rules of ZX-calculus to rewrite the GHZ state as a composition of the spiders Z_2^1 and Z_2^0 (you may again ignore the scalars).

c. Using you result from problem 3 (b), translate your ZX-calculus diagram from part (b) into a circuit diagram. You may want to check your results either using matrix arithmetic or by running your circuit on the IBM quantum computer simulator at `https://quantumexperience.ng.bluemix.net/qstage/`.